# HAWASSA CITY ADMINISTRATION EDUCATION DEPARTMENT

# 2ND SEMESTER COMPENSATION LESSON

# MATHEMATICS FOR GRADE 12 (2012E.C)

## UNIT 6

## THREE DIMENSIONAL GEOMETRY AND VECTORS IN SPACE

## Unit Outcomes:

***After completing this unit, you should be able to:***

- *know methods and procedures for setting up coordinate systems in space. know basic facts about coordinates and their use in determining geometric concepts in space.*
- *apply facts and principles about coordinates in space to solve related problems.*
- *know specific facts about vectors in space.*

## MAIN CONTENTS


**6.1 COORDINATE AXES AND COORDINATE PLANES IN SPACE**

**6.2 COORDINATES OF A POINT IN SPACE**

**6.3 DISTANCE BETWEEN TWO POINTS IN SPACE**

**6.4 MIDPOINT OF A LINE SEGMENT IN SPACE**

**6.5 EQUATION OF SPHERE**

**6.6 VECTORS IN SPACE**

# INTRODUCTION

*In applications of mathematics, certain quantities are determined completely by their magnitude for example, length, mass, area, temperature, and energy. We speak of a length of 5 m or a mass of 3 kg; only one number is needed to describe each of these quantities. Such a quantity is called a* **scalar**. *On the other hand, to describe the displacement of an object, two numbers are required: the* ***magnitude*** *and the* ***direction*** *of the displacement. To describe the velocity of a moving object, we must specify both the* ***speed*** *and the* ***direction*** *of travel. Quantities such as displacement, velocity, acceleration, and force that involve magnitude as well as direction are called* ***directed quantities****. One way to represent such quantities mathematically is through the use of* ***vectors.***

## ▼ Geometric Description of Vectors

*A* **vector** *in the plane is a line segment with an assigned direction. We sketch a vector as shown in Figure 1 with an arrow to specify the direction. We denote this vector by . Point A is the* initial point, *and B is the* terminal point *of the vector* $\overrightarrow{AB}$ . *The length of the line segment AB is called the* magnitude *or* length *of the vector and is denoted by* $|\overrightarrow{AB}|$.

*. We use boldface letters to denote vectors. Thus we write .* $u = \overrightarrow{AB}$

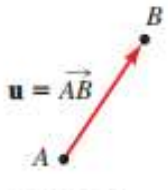
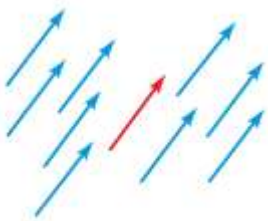


FIGURE 1


FIGURE 2

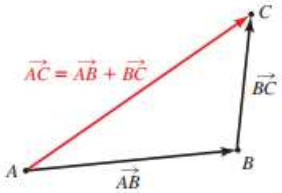
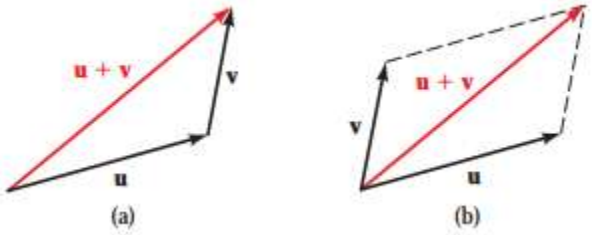


FIGURE 3

*Two vectors are considered* equal *if they have equal magnitude and the same direction. Thus all the vectors in Figure 2 are equal. This definition of equality makes sense if we think of a vector as representing a displacement. Two such displacements are the same if they have equal magnitudes and the same direction. So the vectors in Figure 2 can be thought of as the same displacement applied to objects in different locations in the plane.*

*If the displacement* $u = \overrightarrow{AB}$ *is followed by the displacement* $v = \overrightarrow{BC}$ *, then the resulting displacement* $\overrightarrow{AC}$ *is as shown in Figure 3. In other words, the single displacement represented by the vector* $\overrightarrow{AC}$ *has the same effect as the other two displacements together. We call the vector* $\overrightarrow{AC}$ *the* sum *of the vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{BC}$ *, and we write* $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ *. (The* zero vector, *denoted by* 0, *represents no displacement.) Thus to find the sum of any two vectors* u *and* v, *we sketch vectors equal to* u *and* v *with the initial point of one at the terminal point of the other (see Figure 4(a)). If we draw* u *and* v *starting at the same point, then* u v *is the vector that is the diagonal of the parallelogram formed by* u *and* v *shown in Figure 4(b).*




*If* $a$ *is a real number and* v *is a vector, we define a new vector* $a$v *as follows: The vector* $a$v *has magnitude* $|a||v|$ *and has the same direction as* v *if* $a > 0$ *and the opposite*

*direction if* $a < 0$*. If* $a = 0$*, then* $a$v = 0, *the zero vector. This process is called* multiplication of a vector by a scalar. *Multiplying a vector by a scalar has the effect of stretching or shrinking the vector. Figure 5 shows graphs of the vector* $a$v

*for different values of a. We write the vector* $(-1)v$ *as* $-v$ *. Thus* $-\mathrm{v}$ *is the vector with the same length as* v *but with the opposite direction.*

*The difference of two vectors u and v is defined by u - v = u + ( -v) . Figure 6 shows*

*that the vector u - v is the other diagonal of the parallelogram formed by u and v*




**FIGURE 5** Multiplication of a vector by a scalar

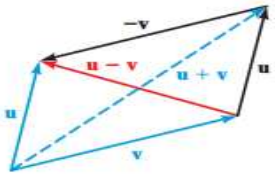



**FIGURE 6** Subtraction of vectors

## ▼ Vectors in the Coordinate Plane

*So far, we've discussed vectors geometrically. By placing a vector in a coordinate plane, we can describe it analytically (that is, by using components). In Figure 7(a), to go from the initial point of the vector v to the terminal point, we move a units to the right and b units upward. We represent v as an ordered pair of real numbers.* $(a\,, b)$

*where a is the* ***horizontal component*** *of* ***v*** *and b is the* ***vertical component*** *of* ***v****. Remember that a vector represents a magnitude and a direction, not a particular arrow in the plane. Thus the vector a, b has many different representations, depending on its initial point (see Figure 7(b)).*

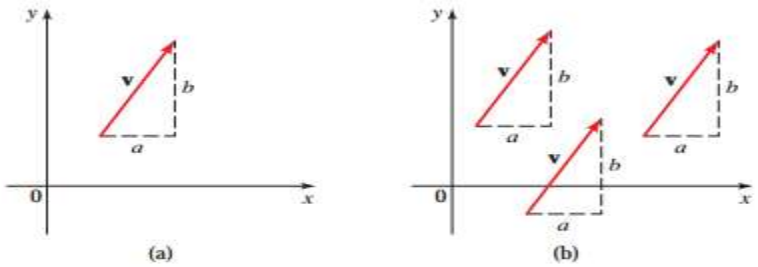



**FIGURE 7**

## COMPONENT FORM OF A VECTOR

If a vector **v** is represented in the plane with initial point $(x_1, y_1)$and terminal point $(x_2, y_2)$ , then $(x_2 - x_1, y_2 - y_1)$

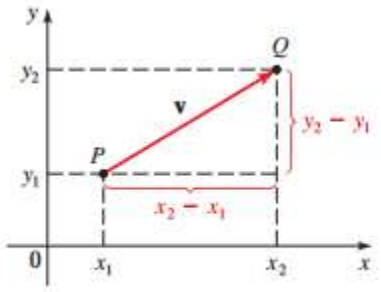


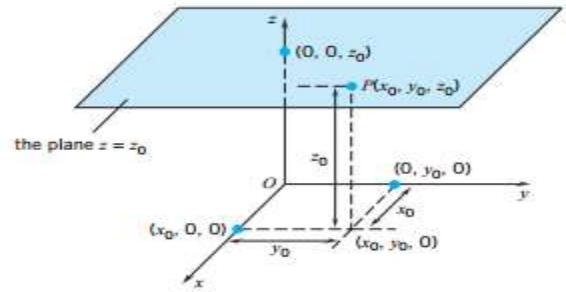

FIGURE 8

# 6.1 COORDINATE AXES AND COORDINATE PLANES IN SPACE

*As in two dimensions, vectors in three-dimensional space have both direction and magnitude. We again visualize vectors as directed line segments joining two points. A vector* ***v*** *is represented by any directed line segment with the appropriate magnitude and direction. We denote the set of all three-dimensional position vectors by*

$$P = \{(x, y, z) |\ x, y, z, \in R\}.$$

*There are now three coordinate planes; the xy-plane, the xz-plane, the yz-plane. A point P in three-dimensional space (see Figure below) is assigned coordinates* $(x_0, y_0, z_0)$ *provided that*



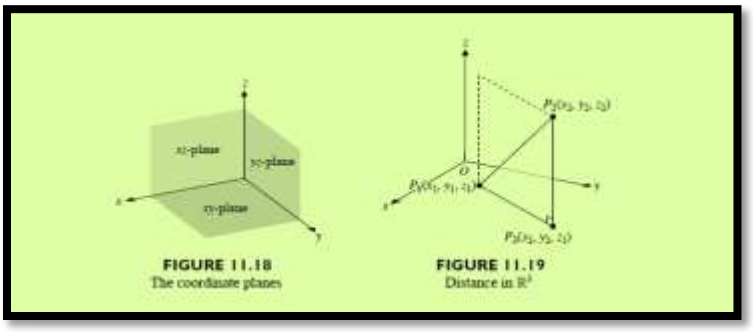




FIGURE 11.18
The coordinate planes

FIGURE 11.19
Distance in $\mathbb{R}^3$

(1) the plane through P which is parallel to the yz-plane intersects the x-axis at$(x_0 , 0 , 0)$, (2) the plane through P which is parallel to the xz-plane intersects the y-axis at $(0 , y_0 , 0)$, (3) the plane through P which is parallel to the xy-plane intersects the z-axis at $(0 , 0 , z_0)$. The coordinates $x_0 , y_0 , z_0$ are called the rectangular coordinates of P.

The plane determined by the x and the y-axes is called the xy-plane, the plane determined by the x and the z-axes is called the xz-plane and the plane determined by the y and the zaxes is called the yz-plane. These three coordinate planes, which intersect at the origin, may be visualized as the floor of a room and two adjacent walls of that room, where the floor represents the xy-plane, the two walls corresponding to xz and yz-planes that intersect on the z-axis and the corner of the room corresponding to the origin.

EXAMPLES '*In each part, find an equation of the sphere with center (2,−1,−3) and satisfying the given condition.*

*(a) Tangent to the xy-plane* *ans'* $(x-2)^2 + (y+1)^2 + (z+3)^2 = 9$

*(b) Tangent to the xz-plane* *Ans;* $(x-2)^2 + (y+1)^2 + (z+3)^2 = 1$

# 6.2 COORDINATES OF A POINT IN SPACE

*The graph of the equation x = a in an xyz-coordinate system consists of all points of the form (a, y, z), where y and z are arbitrary. One such point is (a, 0, 0), and all others are in the plane that passes through this point and is parallel to the yz-plane (Fig- 1). Similarly, the graph of y = b is the plane through (0, b, 0) that is parallel to the xz-plane, (fig-2) and the graph of z = c is the plane through (0, 0, c) that is parallel to the xy-plane. (fig-3)*

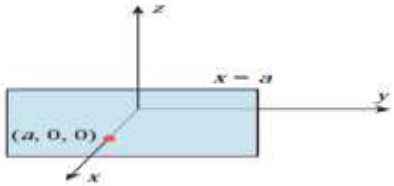



Fig- 1

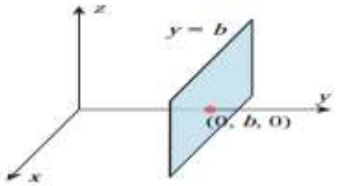



fig-2

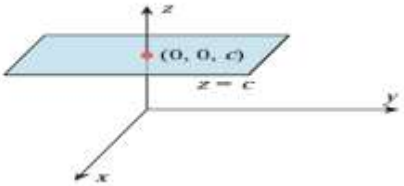



fig-3

*Remember that x represents the directed distance of P from the y-axis and y represents the directed distance of P from the x-axis. For example, if the coordinates of P are ( -3, 2), it means that P is 3 units to the left of the y-axis and 2 units above the x-axis. Similarly, a point Q(4, -5) is found 4 units to the right of the y-axis and 5 units below the x-axis*

*To locate a point in a plane, two numbers are necessary. We know that any point in the Cartesian plane can be represented as an ordered pair* $(a, b)$ *of real numbers, where a is the x-coordinate and b is the y-coordinate. In three-dimensional space, a third dimension is added, so any point in space is represented by an ordered triple* $(a, b, c)$ *of real numbers.*

*To represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the*

***coordinate axes*** *and labeled the x-axis, y-axis, and z-axis. Usually we think of the x- and y-axes as being horizontal and the z-axis as being vertical, and we draw the orientation of the axes as in Figure 1. The three coordinate axes determine the three* ***coordinate planes*** *illustrated in Figure 2(a). The xy-plane is the plane that contains the x- and y-axes; the yz-plane is the plane that contains the y- and z-axes; the xz-plane is the plane that contains the x- and z-axes.*

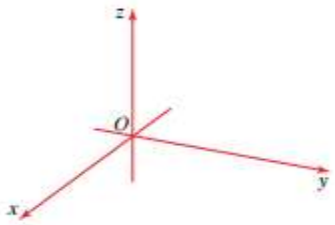



**FIGURE 1** Coordinate axes

These three coordinate planes divide space into eight parts, called **octants**

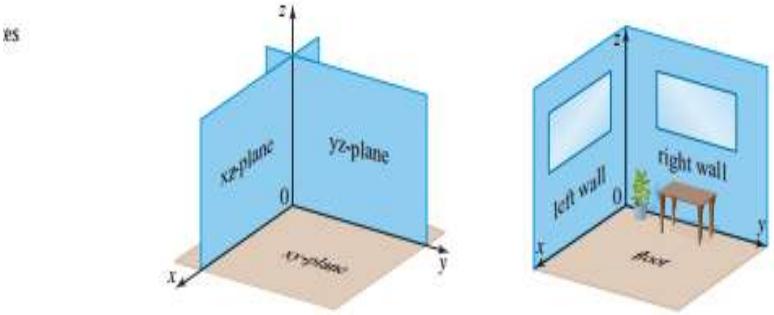


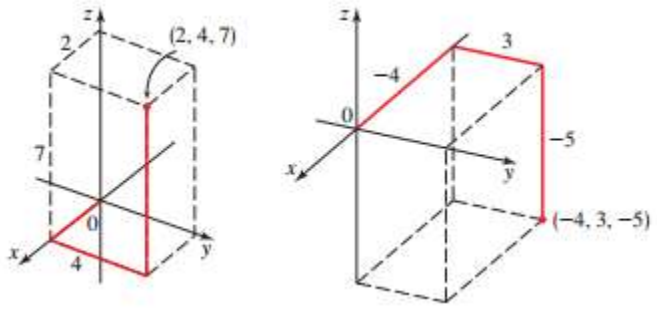

**FIGURE 2** (a) Coordinate planes (b) Coordinate "walls"

*z-axis runs up from the floor toward the ceiling along the intersection of the two walls. Now any point P in space can be located by a unique* ***ordered triple*** *of real numbers* $(a, b, c)$*, as shown in Figure 3. The first number a is the x-coordinate of P, the second number b is the y-coordinate of P, and the third number c is the z-coordinate of P. The set of all ordered triples forms the* ***three-dimensional rectangular coordinate system****.*

**E X A M P L E 1** Plotting Points in Three Dimensions

Plot the points $(2, 4, 7)$ and $(-4, 3, -5)$.

**S O L U T I O N** The points are plotted below




***E X A M P L E*** *Surfaces in Three-Dimensional Space*

*Describe and sketch the surfaces represented by the following equations:*

***(a) z=3 (b) y=5***

***S O L U T I O N***

***(a)*** *The surface consists of the points $(x, y, z)$ where the z-coordinate is 3. This is the horizontal plane that is parallel to the xy-plane and three units above it, as in Figure 5.*

***(b)*** *The surface consists of the points $(x, y, z)$ where the y-coordinate is 5. This is the vertical plane that is parallel to the xz-plane and five units to the right of it, as in Figure 6*

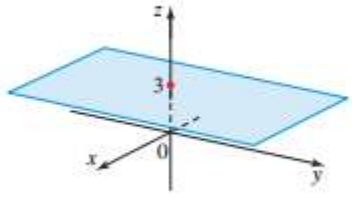



**FIGURE 5** The plane $z = 3$

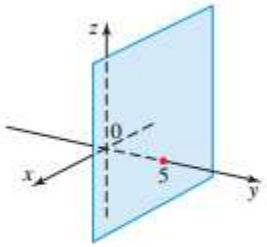



**FIGURE 6** The plane $y = 5$

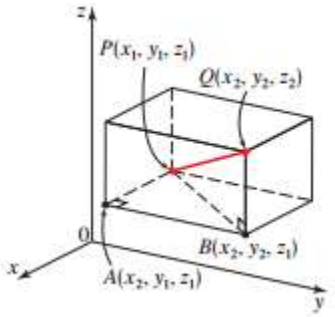
## Distance Formula in Three Dimensions

The familiar formula for the distance between two points in a plane is easily extended to the following three-dimensional formula.

**DISTANCE FORMULA IN THREE DIMENSIONS**

The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$




***EXAMPLE*** *Using the Distance Formula*

*Find the distance between the points*

*a)* $A\,(2\,,-1,7)$ *and* $B(1,-3,\ 5)$ *.b)* $A\,(-3\,,4,-1)$ *and* $B(2,\ \ 5,-4)$

***SOLUTION***

***a)*** $A\,(2\,,-1,7)$ *and* $B(1,-3,\ 5)$

*We use the Distance Formula:*

$$d(A,B) = \sqrt{(1-2)^2+(-3-(-1))^2 + (5-7)^2}$$

$$= \sqrt{1+4+4} = 3$$

**b) (ans;$\sqrt{35}$)**

# MIDPOINT OF A LINE SEGMENT IN SPACE

*The corresponding of the mid point of the line segment having end points* $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$;

Are given $x = \frac{x_1+x_2}{2}$ , $y = \frac{y_1+y_2}{2}$ and $z = \frac{z_1+z_2}{2}$

$$Mp = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Example find the midpoint of the points

a) $A(1, -2, 0)$ *and* $B(3, 5, 4)$

b) $P(4, -1, 2)$ *and* $Q(5, -3, 6)$

SOLUTION

$$Mp = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

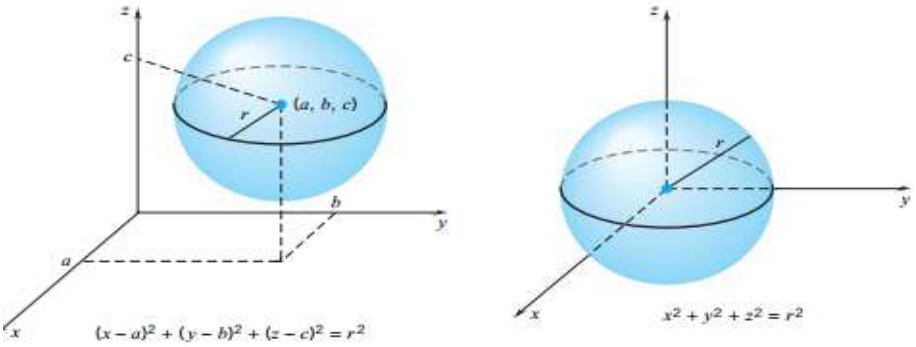
$Mp = \left(\frac{1+3}{2}, \frac{-2+5}{2}, \frac{0+4}{2}\right)$ , which is $\left(2, \frac{3}{2}, 2\right)$

## *EQUATION OF A SPHERE*

*We can use the Distance Formula to find an equation for a sphere in a three-dimensional coordinate space.*

*An equation of a sphere with center* $(a, b, c)$ *and radius r is*

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$




**E X A M P L E ; The equation** $(x-5)^2+(y+2)^2+z^2=9$

Represents the sphere of radius 3 centered at the point $(5-2\,,0)$

Example; Show that the equation $x^2+y^2+z^2+6x+2y-4z=11$

Represents a sphere . Find the center of the sphere and the radius

(ans; $r=5$ and centered $(-3\,,-1\,,2)$

**E X A M P L E ;**

Find an equation of a sphere with

a) radius 5 and center$(-2,1,3)$

b) radius 2 and center$(1,0,-1)$

c) radius 1 and center$(0,1,0)$

**S O L U T I O N** We use the general equation of a sphere, with

$r=5\,,a=-2\;b=1$ and $c=3$

Since an equation of a sphere with center$(a,b,c)$ and radius r is

$(x-a)^2+(y-b)^2+(z-c)^2=r^2$ ,then

: $(x+2)^2+(y-1)^2+(z-3)^2=5^2$

**E X A M P L E** Finding the Center and Radius of a Sphere

Show than $\mathbf{x^2+y^2+z^2+4x-6y+2z+6=0}$ **is** the equation of a sphere, and find its center and radius.

**S O L U T I O N** We complete the squares in the x, y, and z-terms to rewrite the given equation in the form of an equation of a sphere:

than $\mathbf{x^2+4x+y^2-6y+z^2+2z=-6}$

$$\mathbf{x^2+4x+4+y^2-6y+9+z^2+2z+1=-6+14}$$

$$\mathbf{(x+2)^2+(y-3)^2+(z+1)^2=8}$$

Comparing this with the standard equation of a sphere, we can see that the center is

**Center** $\mathbf{(-2,3-1)}$ **and** $\mathbf{r=2\sqrt{2}}$

**E X A M P L E 5** Finding the Trace of a Sphere

Describe the trace of the sphere $\mathbf{(x-2)^2+(y-4)^2+(z-5)^2=36}$

**(a)** the xy-plane and **(b)** the plane $z=9$

**S O L U T I O N**

**(a)** In the xy-plane the z-coordinate is 0. So the trace of the sphere in the xy-plane consists of all the points on the sphere whose z-coordinate is 0. We replace z by 0 in the equation of the sphere and get

$$\mathbf{(x-2)^2+(y-4)^2+(z-5)^2=36}$$

$$\mathbf{(x-2)^2+(y-4)^2+(0-5)^2=36}$$

$$\mathbf{(x-2)^2+(y-4)^2+25=36}$$

$$\mathbf{(x-2)^2+(y-4)^2=11}$$

Thus the trace of the sphere is the circle

$$(\mathbf{x-2})^2+(\mathbf{y-4})^2=\mathbf{11}$$

which is a circle of radius $\sqrt{11}$ that is in the xy-plane, centered at $(2\ ,4\ 0)$

**(b)** The trace of the sphere in the plane $z = 9$ consists of all the points on the sphere whose $z$-coordinate is 9. So we replace $z$ by 9 in the equation of the sphere and get

$$(x-2)^2+(y-4)^2+(9-5)^2=36 \qquad \text{Replace } z \text{ by } 0$$
$$(x-2)^2+(y-4)^2+16=36 \qquad \text{Calculate}$$
$$(x-2)^2+(y-4)^2=20 \qquad \text{Subtract } 16$$

Thus the trace of the sphere is the circle

$$(x-2)^2+(y-4)^2=20, \qquad z=9$$

which is a circle of radius $\sqrt{20}$ that is 9 units above the $xy$-plane, centered at $(2, 4, 9)$ (see Figure 9(b)).

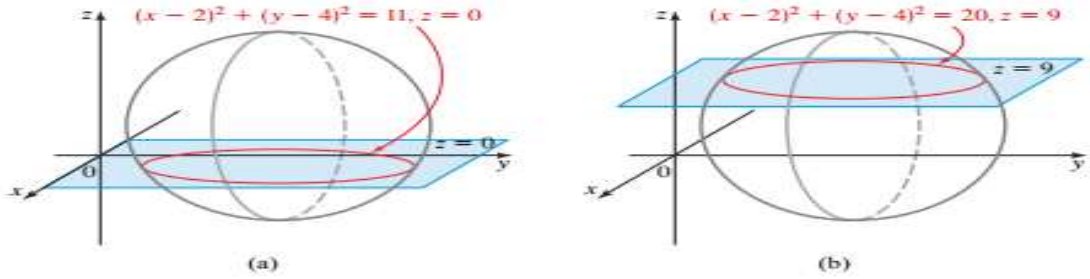



**FIGURE 9** The trace of a sphere in the planes $z = 0$ and $z = 9$

## SELF TESTS

1 *Let* $P=(k,1,-3)$ *and* $Q=(-1,\ k-1,1)$ *are points in* $\mathbb{R}^3$ *and the distance between* $P$ *and* $Q$ *is* $9$ *. Then which one of the following is value of*

*K if* $k>0$ *? [ANS; k=6]*

***2)*** *Show that (2, 1, 6), (4, 7, 9), and (8, 5,−6) are the vertices of a right triangle.*

*(b) Which vertex is at the 90◦ angle?*

*(c) Find the area of the triangle.*

*3 Find the distance from the point (−5, 2,−3) to the*

*(a) xy-plane* *(b) xz-plane* *(c) yz-plane*

*(d) x-axis* *(e) y-axis* *(f ) z-axis*

Suppose the equation of the sphere is given by $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$ .

The which of the following circle is the intersection of the sphere with the yz −plane

A. $(x + 2)^2 + (z - 3)^2 = 6$ C. $(y + 2)^2 + (z - 3)^2 = 25$

B. $(y - 2)^2 + (z - 3)^2 = 10$ D.. $(y + 2)^2 + (z - 3)^2 = 24$

## .REMARKS ‘on’ , inside’ and ‘outside’ the sphete

**In general, if O is the centre of a sphere and r is its radius, then for any point P taken in space, we have one of the following three possibilities.**

**a) OP = r**, in which case P is on the surface of the sphere;

**b) OP < r**, in which case P is inside the sphere; and

**c) OP > r**, in which case P is outside the sphere.

EXAMPLE;

*Decide whether or not each of the following points is inside, outside or on the sphere whose equation is*

*a)* $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$

*b)* $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$

*a)* $(1, 0, 1)$ *b)* $(1,1,1)$ *c)* $(-1, 1, 0)$ *d)* $(2, 1, 0)$

SOLUTION

## SELF TESTS

1. Which of the following is true about the sphere if $(2, -4, 4)$ and $(0, -4, -4)$ are the end points of a diameter of the sphere?

A.. The equation of the sphere is $(x-1)^2 + (y+4)^2 + z^2 = 17$

B. The equation of the sphere is $(x-1)^2 + (y+4)^2 + z^2 = \sqrt{17}$

C. $(1, -4, 0,)$ is a point on of the sphere

D. $(1, 0, 1,)$ is a point outside

3. Which of the following is NOT true about the sphere if $(-2, 0, 4)$ and $(4, 6, -4)$ are the end points of a diameter of the sphere?

A. The equation of the sphere is $(x-1)^2 + (y-3)^2 + z^2 = 34$

B,,. $(1, 0, -5)$ is a point outside of the sphere

C. $(2, 2, 2)$ is a point inside of the sphere

D.$(4, 3, -5)$ is on the sphere

## VECTORS IN THREE DIMENSIONS

*Recall that vectors are used to indicate a quantity that has both magnitude and direction.*

*we studied vectors in the coordinate plane, where the direction is restricted to two dimensions. Vectors in space have a direction that is in three-dimensional space. The properties that hold for vectors in the plane hold for vectors in space as well.*

### ▼ *Vectors in Space*

*Recall that a vector can be described geometrically by its initial point and terminal point. When we place a vector* ***a*** *in space with its initial point at the origin,*

*we can describe it algebraically as an ordered triple:* $(a_1\ , a_2 . a_3 .)$

*where , and are the* ***components*** *of* ***a*** *(see Figure 1). Recall also that a vector has many different representations, depending on its initial point. The following definition gives the relationship between the algebraic and geometric representations of a vector.*

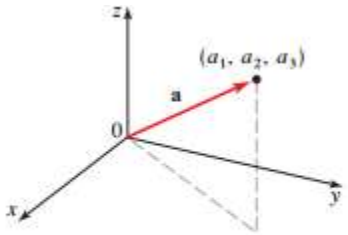



FIGURE 1 $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

*By a vector* ***a*** *we mean an ordered triple of real numbers:* $a = (a_1\ , a_2 . a_3 .)$

*The numbers* $a_1$ , $a_2$ *, and* $a_3$ *are called the components of* ***a****.*

*To say that two vectors are "equal" means that they have the same components;*

$(a_1\ , a_2, a_3 .) = (b_1\ , b_2, b_3 .)$ *iff* $a_1 = b_1,\ \ a_2 = b_2$ *and* $a_3 =. b_3$

*Vectors can be added:* *for* $a = (a_1\ , a_2 . a_3 .)$ *and* $b = (b_1\ , b_2, b_3 .)$

*We define ,* $a + b = (a_1\ , a_2, a_3 .) + (b_1\ , b_2, b_3 .) = (a_1 + b_1\ ,\ a_2 + b_2,\ \ a_3 + b_3\ )$

*By* ***a*** *-* ***b*** *we mean* ***a*** *+ (-****b****); that is ,* $a - b = (a_1\ , a_2, a_3 .) - (b_1\ , b_2, b_3 .)$

$$= (a_1 - b_1\ ,\ a_2 - b_2,\ \ a_3 - b_3\ )$$

*Since the addition of real numbers is commutative and associative, vector addition is*

*commutative and associative:*

$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ , $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$.

*Vectors can be multiplied by scalars (real numbers) for* $a = (a_1 , a_2 . a_3 .)$ *and* $\alpha$ *is* $real$ *we define* $\alpha a = (\alpha a_1 , \alpha a_2 , \alpha a_3 .)$

*Example 1 Given that*

$\boldsymbol{a} = (1, -1, 2)$, $\boldsymbol{b} = (2, 3, -1)$, $\boldsymbol{c} = (8, 7, 1)$, *find*

**(a)** $\boldsymbol{a} - \boldsymbol{b}$. **(c)** $3\boldsymbol{a} - 7\boldsymbol{b}$

**(b)** $2\boldsymbol{a} + \boldsymbol{b}$. . **(d)** $2\boldsymbol{a} + 3\boldsymbol{b} - \boldsymbol{c}$.

*SOLUTION*

*(a) a - b = (1, -1, 2) - (2, 3, -1) = (1 - 2, -1 - 3, 2 + 1) = (-1, -4, 3).*

*(b) 2a + b = 2(1, -1, 2) + (2, 3, -1) = (2, -2, 4) + (2, 3, -1) = (4, 1, 3).*

*(c) 3a - 7b = 3(1, -1, 2) - 7(2, 3, -1)*

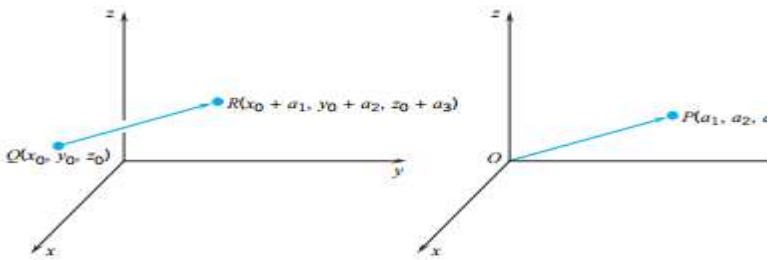
*= (3, -3, 6) - (14, 21, -7) = (-11, -24, 13).*

*(d) 2a + 3b - c = 2(1, -1, 2) + 3(2, 3, -1) - (8, 7, 1)*

*= (2, -2, 4) + (6, 9, -3) - (8, 7, 1) = (0, 0, 0) = 0.*

## *Geometric Interpretation of Vectors*

*Vectors (as we have defined them) have geometric significance only after a coordinate system has been established. In Figure 13.2.1 we have set up a right-handed coordinate system O-xyz. Using that coordinate system, we can represent vectors by arrows in three-dimensional space.*



*To represent the vector* $a = (a_1, a_2. a_3.)$ *, we choose a point* $Q(x_o, y_0, z_0.)$ *as our starting point (we can choose any point) and draw the arrow* $\overrightarrow{QR}$ *to the point* $R(x_0 + a_1, y_0 + a_2, z_0 + a_3)$ *. If we start the arrow at the origin, we use the arrow* $\overrightarrow{OP}$ *with* $P(a_1, a_2. a_3.)$ *.*

*One advantage of using* $\overrightarrow{OP}$ *is that there are fewer coordinates to keep track of. Arrows* $\overrightarrow{QR}$ *and* $\overrightarrow{OP}$ *have different locations, but because they have the same length and the same direction, they represent the same vector: the vector*

$$a = (a_1, a_2. a_3.).$$

*The zero vector* ***0*** *= (0, 0, 0) can be thought of as an arrow of length 0. The zero vector has no direction.*

- *By special convention, all vectors are said to be parallel to* ***0****.*
- *For* $a \neq 0$*, the vectors parallel to* ***a*** *are by definition the scalar multiples* $\alpha$***a.***
- *If* ***a*** *=* ***0****, the scalar multiples of* ***a*** *are all* ***0****.*

## GEOMETRIC INTERPRETATION OF THE DOT PRODUCT

**DEFINITION 13.3.1**

For vectors

$$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k},$$

we define the *dot product* $\mathbf{a} \cdot \mathbf{b}$ by setting

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

**Remark;** For plane vectors $a = a_1 i + a_2 j$ and $b = b_1 i + b_2 j$

$$a.b = (a_1, a_2).(b_1, b_2) = a_1b_1 + a_2b_2$$

For in space vectors $a = a_1i + a_2j + a_3k$ and $b = b_1i + b_2j + b_3k$

$$a.b = (a_1, a_2, a_3.).(b_1, b_2, b_3.)$$

$$a.b = (a_1, a_2, a_3.).(b_1, b_2, b_3.) = (a_1b_1 + a_2b_2 + a_3b_3)$$

Example 1 For a vectors a = 2i - j + 3k, b = -3i + j + 4k, c = i + 3j, we have

a · b = (2)(-3) + (-1)(1) + (3)(4) = -6 - 1 + 12 = 5,
a · c = (2)(1) + (-1)(3) + (3)(0) = 2 - 3 = -1,
b · c = (-3)(1) + (1)(3) + (4)(0) = -3 + 3 = 0.

*Properties of the Dot Product*

*If we dot a vector with itself, we obtain the square of its norm:*

$$a.a = \|a\|^2$$

***i.e*** $\boldsymbol{a.a} = a_1a_1 + a_2a_2 + a_3a_3 = {a_1}^2 + {a_2}^2 + {a_3}^2$

$$a.a = \|a\|^2$$

- *The dot product of any vector with the zero vector is zero:* $a.0 = 0.a$
- *The dot product is commutative:* $a.b = b.a$
- *and scalars can be factored:* $\alpha a.\beta b = \alpha\beta(a.b)$
- *The dot product satisfies the following distributive laws:*

$$\boldsymbol{a \cdot (b + c) = a \cdot b + a \cdot c}\ , \qquad \boldsymbol{(a + b) \cdot c = a \cdot c + b \cdot c}.$$

*Example 2* Given that

$\|a\| = 1$, $\|b\| = 3$ , $\|c\| = 4$ $a.b = 0$ , $a.c = 1$ and $b.c = -2$

find

(a) 3a · (b + 4c).

(b) (a - b) · (2a + b).

(c) [(b · c) a - (a · c)b] · c.

SOLUTION

(a) $3a \cdot (b + 4c) = (3a \cdot b) + (3a \cdot 4c) = 3(a \cdot b) + 12(a \cdot c) = 12.$

(b) $(a - b) \cdot (2a + b) = (a \cdot 2a) + (a \cdot b) + (-b \cdot 2a) + (-b \cdot b)$

$= 2(a \cdot a) + (a \cdot b) - 2(b \cdot a) - (b \cdot b)$

$= 2\|a\|^2 + (a \cdot b) - 2(a \cdot b) - \|b\|^2$

$= 2 + 0 - 2(0) - 9 = -7.$

(c) $[(b \cdot c)a - (a \cdot c)b] \cdot c = [(b \cdot c)a \cdot c] - [(a \cdot c)b \cdot c]$

$= (b \cdot c)(a \cdot c) - (a \cdot c)(b \cdot c) = 0.$

- *By convention, $\theta$, the angle between **a** and **b**, is taken from 0 to $\pi$ (in degrees from 0 to 180); no negative angles. The dot product of two vectors depends on the norms of the vectors and on the angle between them.*
- *For vectors of a given norm,*
- *the dot product measures the extent to which the vectors agree in direction.*
- *As the difference in direction increases, the dot product decreases:*
- *If **a** and **b** have the same direction, then $\theta = 0$ and*

  $a.b = \|a\|\|b\|$ $\quad$ $(\cos 0 = 1)$

  *this is the largest possible value for **a** · **b**.*
- *If **a** and **b** meet at right angles, then $\theta = \frac{\pi}{2}$ and*

  $\boldsymbol{a \cdot b} = 0.$ $\quad$ $\cos\frac{\pi}{2} = 0$

  *If **a** and **b** have opposite directions, then $\theta = \pi$ and*

  $a.b = -\|a\|\|b\|;$ $\quad$ $\cos\pi = -1$

  *this is the least possible value for **a** · **b**.*

- *Two vectors are said to be perpendicular if they lie at right angles to each other or one of the vectors is the zero vector; thus, two vectors are perpendicular iff their dot product is zero.*
- *In symbols* $a \perp b$ *iff* $a.b = 0$

*The unit vectors* ***i, j, k*** *are mutually perpendicular:*

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0.$$

Example 3 Verify that **a** = 2**i** + **j** + **k** and **b** = **i** + **j** - 3**k** are perpendicular.

SOLUTION

**a** · **b** = (2)(1) + (1)(1) + (1)(-3) = 2 + 1 - 3 = 0. ❏

Example 4 Find the value of α for which (3**i** - α**j** + **k**) ⊥ (**i** + 2**j**).

SOLUTION

For the two vectors to be perpendicular, their dot product must be zero.

Since

$$(3\mathbf{i} - \alpha\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j}) = (3)(1) + (-\alpha)(2) + (1)(0) = 3 - 2\alpha,$$

*α must be* $\frac{3}{2}$

## Angle between two vectors in space

**If** $\theta$ is an angle between the two nonzeo vectors $A$ *and* $B$ ,

$$a.b = \|a\|\|b\| \cos\theta$$

*(With **a** and **b** both different from zero)*

*Example* *Given the vectors*

*a)* $A = 6i - 3j + 2k$ *and* $B = 2i + j - 3k$

*b)* $= i - 2j - k$ *and* $B = 5i - 12k$

c) $A = \langle 4, \ 2, \ -6 \rangle$ *and* $B = \langle -5, \ 3, \ -2 \rangle$

*Find* $\cos\theta$

*SOLUTION*

*We first compute* $A.B$ , $\|A\|$ *and* $\|B\|$

$A.B = . \langle 6, -3, \ 2 \rangle . \langle 2 \, , 1, \ -3 \rangle = 12 - 3 - 6 = 3$

$\|A\| = \sqrt{36 + 9 + 4} = \sqrt{49} = 7$ *and* $\|B\| = \sqrt{4 + 1 + 9} = \sqrt{14}$

*Therefore , from* $a.b = \|a\|\|b\| \cos\theta$

$$\cos\theta = \frac{A.B}{\|A\| . \|B\|} = \frac{3}{7\sqrt{14}}$$

$$a.b = \|a\|\|b\| \cos\theta$$

## ***SELF TEST***

*1. Which one of the following is the cosine angle between vectors*

$P = (-1,1,1)$ *and* $Q = (2,2,2)$*?*

$A. \ \frac{4}{3\sqrt{2}}$ $B.. \frac{1}{3}$ $C. \frac{5}{\sqrt{3}}$ $D. \frac{2}{\sqrt{3}}$

2. Let $\vec{a}$ and $\vec{b}$ be two vectors such that $\|\vec{a}\| = 4$ and $\|2\vec{a} - \vec{b}\| = \|\vec{a} + \vec{b}\|$ . Then what is the value of $\vec{a}.\vec{b}$ ?

A. $8\|\vec{b}\|$ B...1/8 C. $4\|\vec{b}\|$ D.1/4