Chapter Four

Introduction to calculus

Chapter Objectives

At the end of this chapter you should be able to:

- become familiar with the concept of limits.
- explain the intuitive meaning of limit of a function.
- evaluate limits of a function at given points.
- identify and evaluate one-sided limits.
- have an understanding of the basic limit theorems.
- acquire basic knowledge on infinite limits and limits at infinity to find asymptotes.
- get acquainted with the concept of continuity of a function.
- apply the intermediate value theorem to locate roots of equations.
- become familiar with the derivative of a function.
- find the slope and equation of a tangent line to a curve.
- get basic knowledge on the techniques of differentiation.
- evaluate the derivative of polynomial, rational and composite functions.
- find the derivatives of the exponential and logarithmic functions.
- develop an appreciation of higher derivatives of functions.
- apply the concepts of the derivative to find rates of change of variable quantities.
- evaluate maximum and minimum values of functions.
- use the concepts of the derivative to sketch the graph of a function.
- get acquainted with related rate problems.
- define an anti-derivative of a continuous function.
- find indefinite integrals of some elementary functions.
- evaluate the integrals of functions using the techniques of substitution, integration by parts and integration by partial fractions.
- solve integrals involving trigonometric functions.
- find the definite integral of continuous functions.
- apply the concepts of definite integrals to find areas of regions bounded by continuous functions.

4.1. Limits and continuity

At the end of this section you should be able to

- become familiar with the concept of limits.
- explain the intuitive meaning of limit of a function.
- evaluate limits of elementary functions at given points.
- identify right-hand limit from left-hand limit.
- evaluate one-sided limits.
- become aware of the relationship between one-sided limits and the existence of limit of a function.
- find limit of a function in terms of its one-sided limits.
- describe the basic limit theorems.
- find limits of functions given in terms of combinations of function.
- evaluate limit of powers of functions.
- evaluate the limit of composite functions.
- apply the squeeze theorem to evaluate limits.
- gain an understanding of the relationship between infinite limits and vertical asymptotes.
- describe horizontal asymptotes in terms of limits at infinity.
- see the relationship between infinite limits at infinity and oblique asymptotes.
- give the definition of continuous function.
- identify the difference between continuous and discontinuous functions.
- state the theorems on continuity.

In this section we study the concepts of limits and continuity of functions. The concept of limit is fundamental to our main subjects of the branch of mathematics called differential and integral calculus. When we ask about the limit of a function at a point c, we are to ask about tendencies of the values of $f(x)$ as x gets arbitrarily closer and closer to c.

Consider the function $f(x) = 2x$ and find values of f for values of x close to 3 (but not necessarily equal to 3).

- values of x to the left of 3

x	2	2.5	2.9	2.99	2.999	…
$f(x)$	4	5	5.8	5.98	5.998	…

- values of x to the right of 3

x	4	3.5	3.1	3.01	3.001	…
$f(x)$	8	7	6.2	6.02	6.002	…

As you can see from the above two tables, the values of $f(x) = 2x$ tend to approach to 6 as x gets closer and closer to 3 from both sides of 3.
Intuitively, we say "6 is the limit of $f(x) = 2x$ as x approaches 3" and we write

$$\lim_{x \to 3}(2x) = 6.$$

In general, if for a given real number c, the values of a function $f(x)$ approaches a number L as x gets close to c, we write

$$\lim_{x \to c} f(x) = L$$

We may sometimes write this as $f(x) \to L$ as $x \to c$.

Suppose f is a function and c is a fixed real number. When one ask for the behavior (approximate value) of $f(x)$ for x near c, normally one is not interested about the value $f(c)$. Instead, one is asking about values of f at $x \in (c - \delta, c + \delta)$ for $x \neq c$, with $\delta > 0$ *(δ - delta).* We call the interval $(c - \delta, c + \delta)$ a ***neighborhood of c***. When we exclude c from the neighborhood, we obtain a union of two disjoint intervals.

i.e., $(c - \delta, c) \cup (c, c + \delta)$.

Such a set is called a ***deleted neighborhood of c***. For $\delta > 0$, the interval $(c - \delta, c)$ may be called a ***left neighborhood of c*** while $(c, c + \delta)$ a ***right neighborhood of c***. Thus when we talk of f ***near*** c, we are interested in the function values ***only*** in a deleted neighborhood of c.

Therefore, when our interest is to know limit of f at c, we are mainly curious to know about the tendencies of $f(x)$ for x in a deleted neighborhood of c.

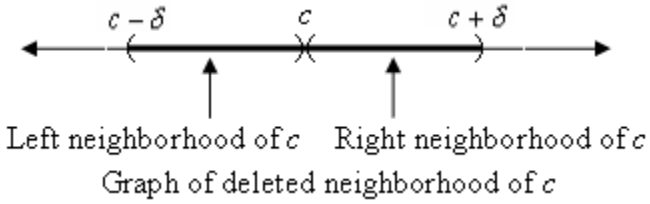


Figure 4.1: Deleted neighborhood of c

Similarly, if x gets close to 2, the function $f(x) = x + 3$ gets close to 5, so that $\lim_{x\to 2}(x+3) = 5$

and if x gets close to 1, $f(x) = x^2 - 3$ approaches –2, so that $\lim_{x\to 1}(x^2 - 3) = -2$.

You can also see that

$$\lim_{x\to 2}(x^3+1) = 9, \qquad \lim_{x\to 8}\sqrt{x+1} = 3, \quad \text{and} \quad \lim_{x\to 1}\frac{1}{x-5} = -\frac{1}{4}$$

In the above examples we were able to find the limits without much difficulty. However, finding certain limits are not so immediate. For example consider

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

Here both $x^2 - 4$ and $x - 2$ approach to 0 as x approaches to 2, and $0/0$ is not determined. But note that

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x + 2, \quad \text{for } x \neq 2.$$

Thus, for x close to 2 (but not necessarily equal to 2), the behavior of $\frac{x^2-4}{x-2}$ is similar to that of $x + 2$ and it seems reasonable to conclude that

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

In the same manner, we have

$$\lim_{x\to 0}\frac{(x+2)^2-4}{x} = \lim_{x\to 0}\frac{x^2+4x+4-4}{x} = \lim_{x\to 0}\frac{x(x+4)}{x} = \lim_{x\to 0}(x+4) = 4, \text{ (for } x \neq 0)$$

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}, \frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x\to 1}\frac{x-1}{(x-1)(\sqrt{x}+1)} = \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{\sqrt{x}+1} = \frac{1}{2},$$

(for $x \neq 1$)

Even though standard textbooks of calculus give the formal (analytic) definition of limit of a function using the notion of neighborhoods, we shall give here a working definition in terms of what we call ***one-sided limits***.

Definition 4.1:
Suppose f is a function and c is a fixed real number.

1. A real number L is called the **left-hand limit** of f at c, written as $\lim_{x\to c^-} f(x) = L$

if and only if for all values of x sufficiently close to c from the **left side** of c, the corresponding values of f approach to L.

2. A real number R is called the **right-hand limit** of f at c, written as $\lim_{x \to c^+} f(x) = R$

if and only if for all values of x sufficiently close to c from the **right side** of c, the corresponding values of f approach to R.

Note that, if the set $(c - \delta, c) \cup (c, c + \delta)$ is a deleted neighborhood of c, then for left-hand limit we take $x \in (c - \delta, c)$, i.e. $x < c$, and for right-hand limit we take $x \in (c, c + \delta)$, i.e. $x > c$ (but not necessarily $x = c$).

Example 4.1: Let $f(x) = \begin{cases} x^2, & for\ x < 1 \\ 2x, & for\ x > 1 \end{cases}$ Then $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1$, while $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x = 2$

Example 4.2: Let $f(x) = \frac{|x|}{x} = \begin{cases} -1, & for\ x < 0 \\ 1, & for\ x > 0 \end{cases}$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 1^-} (-1) = -1$

and $\lim_{x \to 0^+} f(x) = 1$

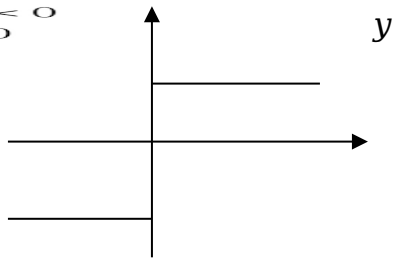
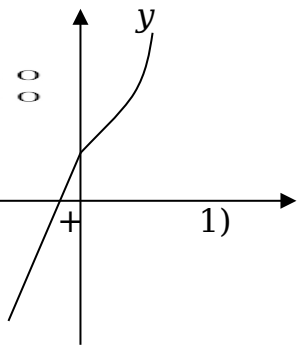


Figure 4.1

Example 4.3 : Let $f(x) = \begin{cases} 2x + 1, & for\ x < 0 \\ x^2 + 1, & for\ x > 0 \end{cases}$



The $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} (2x + 1) = 1$

and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 + 1) = 1$

Figure 4.2

Note that in this example $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

Definition 4.2:

Suppose f is function and c is a fixed real number. A real number L is called the **limit of f at *c*** if and only if the left-and right-hand limits exist and are both equal to L;

i.e. $\lim_{x\to c} f(x)$ = L if and only if $\lim_{x\to c^-} f(x)$ = L = $\lim_{x\to c^+} f(x)$

Thus for $\lim_{x\to c} f(x)$ *to exist, the following conditions must be satisfied:*

i) $\lim_{x\to c^-} f(x)$ *must exist*

ii) $\lim_{x\to c^+} f(x)$ *must exist*

iii) $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$

Otherwise, we say $\lim_{x\to c} f(x)$ **does not exist.**

Thus in **Example 4.3**, where $f(x) = \begin{cases} 2x+1, & for\ x<0 \\ x^2+1, & for\ x>0 \end{cases}$

we have seen above that $\lim_{x\to 0^-} f(x) = 1 = \lim_{x\to 0^+} f(x)$ Thus, $\lim_{x\to 0} f(x) = 1$.

Example 4.4: Let $f(x) = 2^x$ for $x \in \Re$. Then

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2^x = 2^1 = 2 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2^x = 2^1 = 2$$

Since $\lim_{x\to 0^-} 2^x = \lim_{x\to 1^+} 2^x = 2$, we have $\lim_{x\to 1} 2^x = 2$

In fact, if $a > 0$, $a \neq 1$, then $\lim_{x\to c} a^x = a^c$, for any $c \in \mathbf{R}$

Similarly, you can show that $\lim_{x\to c} \log_a^x = \log_a^x = \log_a^c$, for c > 0 and $\lim_{x\to c} \sin x = \sin c$, $\forall c$.

Example 4.5: Let f(x) = $\begin{cases} x^2, & for\ x \leq 1 \\ 3, & for\ x > 1 \end{cases}$

Then $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$ while $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3 = 3$. Since $1 \neq 3$, $\lim_{x\to 1} f(x)$ does not exist.

Example 4.6: Let $f(x) = \sqrt{x}$, for $x \geq 0$. Then $\lim_{x\to 0^+} f(x) = 0$. But since $f(x) = \sqrt{x}$ is not defined to the left of 0, $\lim_{x\to 0^-} \sqrt{x}$ does not exist. Hence $\lim_{x\to 0} \sqrt{x}$ does not exist.

Remark: *If a function f has a limit as x approaches a number c, then the limit is* ***unique****; i.e.*

if $\lim_{x\to c} f(x) = L_1$ *and* $\lim_{x\to c} f(x) = L_2$, *then* $L_1 = L_2$.

- **Basic Limit Theorems**

Theorem 4.1: Suppose $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$ and k is a constant.

Then i) $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x) = KL$ … **Constant Rule**

ii) $\lim_{x\to c}(f + g)(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M$ … **Addition Rule**

iii) $\lim_{x\to c}(f - g)(x) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L - M$ … **Difference Rule**

Example 4.7: $\lim_{x\to \pi/2} 5\sin x = 5\lim_{x\to \pi/2} \sin x = 5(1) = 5$

Example 4.8: Let $f(x) = 2x$ and $g(x) = 5x - 1$. Then

$\lim_{x\to 1}(f + g)(x) = \lim_{x\to 1} f(x) + \lim_{x\to 1} g(x) = \lim_{x\to 1}(2x) + \lim_{x\to 1}(5x - 1) = 2(1) + 5(1) - 1 = 2 + 5 - 1 = 6$

$\lim_{x\to 3}(f - g)(x) = \lim_{x\to 3} f(x) - \lim_{x\to 3} g(x) = \lim_{x\to 3}(2x) - \lim_{x\to 3}(5x - 1) = 2(3) - [5(3) - 1] = 6 - 14 = -8$

Theorem 4.2: Assume that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$.

Then $\lim_{x\to c}(fg)(x) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right) = L.M$ … **Product Rule**

Example 4.9: $\lim_{x\to \pi} x\cos x = \lim_{x\to \pi} x.\lim_{x\to \pi} \cos x = \pi.\cos\pi = \pi(-1) = -\pi.$

It follows from Theorem 4.2 that $\lim_{x\to c} x^2 = \lim_{x\to c} x.x = \lim_{x\to c} x.\lim_{x\to c} x = c.c = c^2$

In general, if n is a positive integer, $\lim_{x\to c} x^n = c^n$.

Thus, if $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ is any polynomial function of degree n and c is any real number, then from Theorems 4.1 and 4.2, we get

$$\lim_{x\to c} P(x) = \lim_{x\to c}(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0)$$
$$= \lim_{x\to c} a_n x^n + \lim_{x\to c} a_{n-1}x^{n-1} + \ldots + \lim_{x\to c} a_2x^2 + \lim_{x\to c} a_1x + \lim_{x\to c} a_0$$
$$= a_n\lim_{x\to c} x^n + a_{n-1}\lim_{x\to c} x^{n-1} + \ldots + a_2\lim_{x\to c} x^2 + a_1\lim_{x\to c} x + a_0$$
$$= a_nc^n + a_{n-1}c^{n-1} + \ldots + a_2c^2 + a_1c + a_0 = P(c)$$

Example 4.10: Let $P(x) = 2x^3 + 4x^2 - 3x + 1$. Then

$\lim_{x\to -1} P(x) = \lim_{x\to -1}(2x^3 + 4x^2 - 3x + 1) = 2(-1)^3 + 4(-1)^2 - 3(-1) + 1 = -2 + 4 + 3 + 1 = 6$

Theorem 4.3: Assume that $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$ and suppose $M \neq 0$

Then $\lim_{x\to c}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}$ ... **Quotient Rule**

Example 4.11: $\lim_{x\to 10}\frac{\log x}{x} = \frac{\lim_{x\to 10}\log x}{\lim_{x\to 10} x} = \frac{1}{10}$.

If $f(x) = \frac{p(x)}{q(x)}$ *is a rational function, then* $\lim_{x\to c} f(x) = \lim_{x\to c}\frac{p(x)}{q(x)} = \frac{\lim_{x\to c} p(x)}{\lim_{x\to c} q(x)} = \frac{p(c)}{q(c)} = f(c)$ if $f(c) \neq 0$.

Example 4.12: $\lim_{x\to -2}\frac{x^3 - 4x + 1}{4x^2 + x - 6} = \frac{\lim_{x\to -2}(x^3 - 4x + 1)}{\lim_{x\to -2}(4x^2 + x - 6)} = \frac{(-2)^3 - 4(-2) + 1}{4(-2)^2 + (-2) - 6} = \frac{-8 + 8 + 1}{16 - 2 - 6} = \frac{1}{8}$.

Theorem 4.4: Suppose $\lim_{x\to c} f(x) = L$, $L \neq 0$ and $a \in \mathbf{R}$ such that $L^a \in \mathbf{R}$

Then, $\lim_{x\to c}(f(x))^a = L^a$ ..... **Power Rule**

Example 4.13: $\lim_{x\to \frac{\pi}{2}}\sqrt{\sin x} = \lim_{x\to \frac{\pi}{2}}(\sin x)^{1/2} = \left[\lim_{x\to \frac{\pi}{2}}(\sin x)\right]^{1/2} = 1^{\frac{1}{2}} = \sqrt{1} = 1$.

Example 4.14: $\lim_{x\to 4}\sqrt[3]{(x^2 + 2x + 3)^2} = \lim_{x\to 4}(x^2 + 2x + 3)^{2/3} = (16 + 4 + 3)^{2/3} = (27)^{2/3} = 3^2 = 9$

Theorem 4.5 (The Squeezing Theorem). Suppose f, g and h are functions such that $f(x) \leq h(x) \leq g(x)$ for all x in some deleted neighborhood

of c. If $\lim_{x\to c} f(x) = L = \lim_{x\to c} g(x)$, then $\lim_{x\to c} h(x) = L$.

Example 4.14: Evaluate $\lim_{x\to 0} x^2 \sin\frac{1}{x}$

Solution: It may be tempting to consider $x^2 \sin\frac{1}{x}$ as the product of x^2 and $\sin\frac{1}{x}$ and then use the Product Rule. Unfortunately it can be shown that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist. Thus we cannot use the Product Rule to evaluate the given limit. However since the sine function has range [–1, 1], it follows that $-1 \le \sin\frac{1}{x} \le 1$, for $x \neq 0$. Multiplying both sides by x^2, we get

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2 \text{ with } \lim_{x\to 0}(-x^2) = 0 = \lim_{x\to 0} x^2$$

Thus, by the Squeeze Theorem, we get $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

Remark: One of the most important applications of the Squeezing Theorem is evaluating $\lim_{x\to 0}\frac{\sin x}{x}$. We cannot apply the Quotient Rule to evaluate this limit since the limit of the denominator is 0. But using some geometric constructions and *the Squeeze Theorem* it can be shown that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Remark: *The above result has important consequences especially in the evaluation of some limits involving trigonometric functions.*

Example 4.15: Find $\lim_{x\to 0}\frac{\sin 5x}{x}$

Solution: $\lim_{x\to 0}\frac{\sin 5x}{x} = \lim_{x\to 0}\frac{5\sin 5x}{5x} = 5\lim_{x\to 0}\frac{\sin 5x}{5x}$

If we put y = 5x, we have as x → 0, 5x → 0 so that y → 0. Thus $\lim_{x\to 0}\frac{\sin 5x}{x} = 5$

$\lim_{y\to 0}\frac{\sin y}{y} = 5$.

In general, for any $a \in \Re$, $\lim_{x\to 0}\frac{\sin ax}{x} = a$.

Example 4.16: Find $\lim_{x\to 0}\frac{\tan x}{x}$

Solution: $\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\left(\frac{\sin x}{\cos x}.\frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cdot\lim_{x\to 0}\left(\frac{1}{\cos x}\right) = 1.\frac{1}{1} = 1.$

Example 4.17: Evaluate $\lim_{x\to 0}\frac{\cos x - 1}{x}$

Solution: By multiplying both numerator and denominator by cos x + 1 we get

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = \lim_{x\to 0}\left(\frac{\cos x - 1}{x}\right)\left(\frac{\cos x + 1}{\cos x + 1}\right) = \lim_{x\to 0}\frac{\cos^2 x - 1}{x(\cos x + 1)}$$

$$= \lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x + 1)} \qquad \text{(since } \sin^2 x + \cos^2 x = 1\text{)}$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{-\sin x}{\cos x + 1}\right) = \lim_{x\to 0}\frac{\sin x}{x}.\lim_{x\to 0}\frac{-\sin x}{\cos x + 1} = 1.\frac{0}{1} = 1.0 =$$

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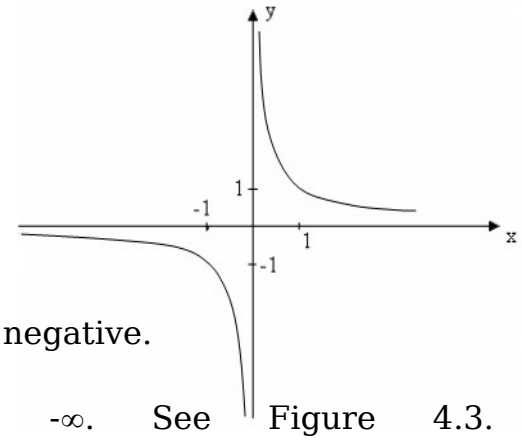
- **Infinite Limits, Limits at Infinity and Asymptotes**

When $\lim_{x\to c^+} f(x)$ does not exist, it may happen that as x approaches *c* from right, the value of f(x) becomes indefinitely large or becomes negative and indefinitely large in absolute value. The value of f(x) may behave similarly when the left-hand limit at *c* does not exist. We shall use the symbols ∞ (infinity) and $-\infty$ to express these cases, respectively.

To explain these concepts consider the function f(x) = $\frac{1}{x}$, for x ≠ 0

As x gets close to 0 from right, the values of f(x) = $\frac{1}{x}$ become arbitrarily large positive.

In this case we write $\lim_{x\to 0^+}\frac{1}{x} = \infty$



and when x gets close to 0 from left,

the values of f(x) = $\frac{1}{x}$ become arbitrarily small negative.

In this case we write $\lim_{x\to 0^-}\frac{1}{x} = -\infty$. See Figure 4.3.

Figure 4.3

Definition 4.3:

Let f be a function defined in a deleted neighborhood of c.

i) We say that the **left-hand limit** of f(x) at c is **infinity**, and write $\lim_{x\to c^-}$ f(x) = ∞

if for every real number M, we have f(x) > M for every x close to c from the left side of

ii) We say that the **right-hand limit** of f(x) at c is **infinity**, and write $\lim_{x\to c^+}$f(x) = ∞

if for every real number M, we have f(x) > M for every x close to c from the right side if c.

iii) We say that the **limit** of f(x) at c is **infinity** and write $\lim_{x\to c}$f(x) = ∞

if and only if $\lim_{x\to c^-}$f(x) = ∞ and $\lim_{x\to c^+}$f(x) = ∞

Definition 4.4:

Let f be a function defined in a deleted neighborhood of c.

i) We say that the **left-hand limit** of f(x) at c is **negative infinity**, and write $\lim_{x\to c^-}$f(x) = -∞

if for every real number M, we have f(x) < M for every x close to c from the left side of c.

ii) We say that the **right-hand limit** of f(x) at c is **negative infinity**, and write $\lim_{x\to c^+}$f(x) = -∞

if for every real number M, we have f(x) < M for every x close to c from the right side if c.

iii) We say that the **limit** of f(x) at c is **negative infinity** and write $\lim_{x\to c}$f(x) = -∞

if and only if $\lim_{x\to c^-}$f(x) = -∞ and $\lim_{x\to c^+}$f(x) = -∞

Example 4.18: For f(x) = $\frac{1}{x}$, for x ≠ 0, $\lim_{x\to 0^+}\frac{1}{x}$ = ∞ and $\lim_{x\to 0}\frac{1}{x}$ = -∞. Hence $\lim_{x\to 0}\frac{1}{x}$ does not exist. Whereas for f(x) = $\frac{1}{x^2}$, x ≠ 0, $\lim_{x\to 0^+}\frac{1}{x^2}$ = ∞ = $\lim_{x\to 0^-}\frac{1}{x^2}$.

Hence $\lim_{x\to 0}\frac{1}{x^2}$ = ∞.

In general, for any real number c and f(x) = $\frac{1}{x-c}$ we have $\lim_{x\to c^+}$f(x) = $\lim_{x\to c^+}\frac{1}{x-c}$ = ∞ and

$\lim_{x\to c^-} f(x) = \lim_{x\to c^-} \frac{1}{x - c} = -\infty$.

Definition 4.5:
Suppose f is a function and c is a fixed real number. We say that the line x = c is a **vertical asymptote** of the graph of f if and only if either

$$\lim_{x\to c^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to c^+} f(x) = \pm\infty$$

Remark: *From the above examples, we can see that the line x = 0 (i.e. the y-axis) is a vertical asymptote of the graphs of the functions f(x) =* $\frac{1}{x}$ *and f(x) =* $\frac{1}{x^2}$*, while the line x = c is a vertical asymptote of the graph of f(x) =* $\frac{1}{x - c}$.

Example 4.19: Find all the vertical asymptotes of f(x) = $\frac{x+2}{x^2 - 1}$

Solution: If c is any number different from 1 or -1, then by the Quotient Rule,

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{x+2}{x^2 - 1} = \frac{c+2}{c^2 - 1} \in \mathbf{R}$$

Thus any line x = c for c ≠ ± 1 cannot be a vertical asymptote.

For c = 1, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x+2}{x^2 - 1} = \lim_{x\to 1^+} \left(\frac{x+2}{x+1}\right)\left(\frac{1}{x - 1}\right)$

$$= \lim_{x\to 1^+} \left(\frac{x+2}{x+1}\right).\lim_{x\to 1^+} \left(\frac{1}{x - 1}\right) = \frac{3}{2}(\infty) = \infty$$

Similarly, for c = -1, $\lim_{x\to 1^+} f(x) = \lim_{x\to -1^+} \left(\frac{x+2}{x - 1}\right).\left(\frac{1}{x+1}\right)$

$$= \lim_{x\to -1^+} \left(\frac{x+2}{x - 1}\right).\lim_{x\to 1^+} \left(\frac{1}{x+1}\right) = \left(-\frac{1}{2}\right)(\infty) = -\infty$$

Hence the lines x = 1 and x = -1 are vertical asymptotes of the graph of the function f(x) = $\frac{x+2}{x^2 - 1}$.

Next, we try to investigate the behavior of a function f as x increases (or decreases) indefinitely, and try to see if we have $\lim_{x\to\infty} f(x)$ or $\lim_{x\to -\infty} f(x)$. Such limits, if they exist, are in general called limits at infinite.

Definition 4.6:

i) Suppose f is a function defined on an interval of the form (c, ∞), for some c∈**R**. We say that the limit of f(x) **as x approaches to infinity** is the number L, and write $\lim_{x\to\infty} f(x) = L$ if when

x is assigned sufficiently large positive values, the corresponding values of f approach to L.

ii) Suppose f is a function defined on an interval of the form $(-\infty, c)$ for some $c \in \mathbf{R}$. We say that the limit of f(x) as **x approaches to negative infinity** is the number L, and write $\lim_{x \to -\infty} f(x) = L$

if when x is assigned sufficiently small negative values, the corresponding values of f approach to L.

Example 4.20: Let $f(x) = \frac{1}{x}$, for $x \neq 0$.

When x is assigned sufficiently large positive values, the values of $f(x) = \frac{1}{x}$ become close to 0. Similarly for values of x sufficiently small negative values, $f(x) = \frac{1}{x}$ becomes close to 0. Hence $\lim_{x \to \infty} \frac{1}{x} = 0$ and $\lim_{x \to -\infty} \frac{1}{x} = 0$. See Figure 4.3 above.

Similarly, $\lim_{x \to \infty} \frac{1}{x^2} = 0$, $\lim_{x \to -\infty} \frac{1}{x^2} = 0$ and in general, $\lim_{x \to \infty} \frac{1}{(x-c)^2} = 0$.

Definition 4.7:

If for a function f and a real number L, $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, then the line y = L is called a **horizontal asymptote** to the graph of f.

Thus the line y = 0 (i.e. the x-axis) is a horizontal asymptote for both the function $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$. See Figure 4.3 above.

Example 4.21: Find a horizontal asymptote to the graph of $f(x) = \frac{3x^2 - x + 1}{2x^2 + 5}$

Solution: Since we are interested with the behavior of f for large values of |x|, we divide both numerator and denominator of f by the leading exponent (i.e.x^2) to get

$$f(x) = \frac{3x^2 - x + 1}{2x^2 + 5} = \frac{\frac{3x^2}{x^2} - \frac{x}{x^2} + \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{5}{x^2}} = \frac{3 - \frac{1}{x} + \frac{1}{x^2}}{2 + \frac{5}{x^2}}$$

Then $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2 - x + 1}{2x^2 + 5} = \frac{\lim_{x\to\infty}\left(3 - \frac{1}{x} + \frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(2 + \frac{5}{x^2}\right)} = \frac{3 - 0 - 0}{2 + 0} = \frac{3}{2}$

Thus $\lim_{x\to\infty} f(x) = \frac{3}{2}$ and the line $y = \frac{3}{2}$ is a horizontal asymptote to the graph of f.

Similarly, $\lim_{x\to -\infty} f(x) = \frac{3}{2}$.

Remark: *For a rational function* $f(x) = \frac{p(x)}{q(x)}$, *with deg(p)<deg(q), we find a horizontal asymptote by applying the above technique.*

As a combination of the above two subsections, it may happen that as the values of |x| increase without bound, the corresponding values of |f(x)| also increases without bound leading to what are generally called infinite limits at infinity.

Definition 4.8:
Let f be defined on an interval of the form (c, ∞), for c∈**R**. We say that the limit of f(x) as **x approaches to infinity is infinity**, written $\lim_{x\to\infty} f(x) = \infty$ whenever x is assigned sufficiently large positive values, the corresponding values of f(x) increase without bound.

Remark: *Analogous definitions can be given for* $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to -\infty} f(x) = \infty$ *and* $\lim_{x\to -\infty} f(x) = -\infty$.

Example 4,22: For $f(x) = x^3$, we have $\lim_{x\to\infty} x^3 = \infty$ and $\lim_{x\to -\infty} x^3 = -\infty$

Example 4.23: $\lim_{x\to\infty} \frac{x^4 - 3x + 2}{2x^2 + 1} = \lim_{x\to\infty} \frac{1 - \frac{3}{x^3} + \frac{2}{x^4}}{\frac{2}{x^2} + \frac{1}{x^4}} = \infty$. (By dividing by x^4).

Definition 4.9:
If for a function f and for two real numbers a and b $\lim_{x\to\pm\infty} [f(x) - (ax + b)] = 0$, then the line
y = ax + b is called an **oblique** (or a **skew**) asymptote to the graph of f.

In general, for a rational function $f(x) = \frac{p(x)}{q(x)}$, *we have*

i) When degree(p) < degree (q), $\lim_{x\to\pm\infty} f(x) = 0$ and the x-axis is a horizontal asymptote of f.

ii) When degree(p) = degree(q), then f has a horizontal asymptote given by the quotient of the leading coefficients of p and q.

iii) When degree(p) > degree(q), then $\lim_{x\to\pm\infty} f(x) = \pm\infty$, and in particularly if degree(p) = degree(q)+1, then f has an oblique asymptote obtained as a quotient when we divide p by q.

Example 4.24: Let f(x) = $\frac{4x^2+5x-1}{x+3}$, find all asymptotes of f.

Solution: Since $\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \frac{4x^2+5x-1}{x+3} = \infty$, the line x = -3 is a vertical asymptote.

By the long division method, we get f(x) = $\frac{4x^2+5x-1}{x+3}$ = (4x – 7) + $\frac{20}{x+3}$

$\Rightarrow \lim_{x\to\infty}$[f(x) – (4x – 7)] = $\lim_{x\to\infty} \frac{20}{x+3} = 0$

Therefore, the line y = 4x – 7 is an oblique asymptote of f.

- **A special Limit in Exponential Function**

Consider the function f(x) = $\left(1+\frac{1}{x}\right)^x$ with domain $(-\infty, -1) \cup (0, \infty)$

The following two tables indicate the behavior of the values of f(x) as x approaches to positive and negative infinity, respectively,

x	2	10	100	1000	10,000	100,000
$f(x)$	2.75	2.593743	2.704814	2.716924	2.718146	2.718268

x	-2	-10	-100	-1000	-10,000	-100,000
$f(x)$	4	2.867972	2.731999	2.719642	2.718418	2.718295

As is tried to be indicated from the above tables, the values of $\left(1+\frac{1}{x}\right)^x$ tend to approach to an irrational number whose value is 2.7182818…. This number,

denoted by e, is called the **base of the natural logarithm**, and plays an important role in calculus.

Remark: *The natural logarithmic function (with base e) is given by f(x) =* $\log_e x$ *and is denoted by f(x) =* ℓnx*. Its inverse, the* ***natural exponential function*** *is given by f(x) = exp(x) =* e^x.

Thus from the above constructions, we have

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = \lim_{x\to -\infty}\left(1+\frac{1}{x}\right)^x.$$

This limit has important consequences.

Example 4.25: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+3} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.\left(1+\frac{1}{x}\right)^3$ - Rule of exponents

$= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x . \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^3$ - Product Rule

$= e.1^3 = e$

In general, for any real number *a*, $\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^{x+a} = e$.

Example 4.26: Show that $\lim_{t\to 0}(1+t)^{1/t} = e$

Solution: We prove this by showing that $\lim_{t\to 0^+}(1+t)^{1/t} = e = \lim_{t\to 0^-}(1+t)^{1/t}$. First use the substitution t = $\frac{1}{x}$, so that *x* = $\frac{1}{t}$ and as $x \to \infty$, $t \to 0^+$. Hence, $\lim_{t\to 0^+}(1+t)^{1/t} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

Similarly, as $x \to -\infty$, $t \to 0^-$. Hence, $\lim_{t\to 0^-}(1+t)^{1/t} = \lim_{x\to -\infty}\left(1+\frac{1}{x}\right)^x = e$

Therefore, $\lim_{t\to 0^+}(1+t)^{1/t} = e = \lim_{t\to 0^-}(1+t)^{1/t} \Rightarrow \lim_{t\to 0}(1+t)^{1/t} = e$.

Example 4.27: Evaluate $\lim_{x\to\infty}\left(1-\frac{5}{x}\right)^x$

Solution: Let t = $\frac{-5}{x}$. Then x = $\frac{-5}{t}$ and

$$\lim_{x\to\infty}\left(1-\frac{5}{x}\right)^x = \lim_{t\to 0^-}(1+t)^{-5/t} = \lim_{t\to 0^-}\left[(1+t)^{1/t}\right]^{-5} = \left[\lim_{t\to 0}(1+t)^{1/t}\right]^{-5} = e^{-5} = \frac{1}{e^5}.$$

In general, for any real number *a*, $\lim_{x\to\pm\infty}\left(1+\frac{a}{x}\right)^x = e^a$.

- **Continuity of a Function**

In our everyday usage, the word continuity refers to something that happens without any interruption. In calculus, the term continuity is used to describe functions whose graphs can be traced without any break. We shall give its formal definition using the concept of limits.

Definition 4.10:

- Let f be a function and c be a number in the domain of f. f is said to be **continuous at c** if

$$\lim_{t \to c} f(x) = f(c)$$

- If f fails to be continuous at c, then we say that f is **discontinuous (**or **not continuous) at c.**
- f is said to **continuous** if it is continuous at each point of its domain.

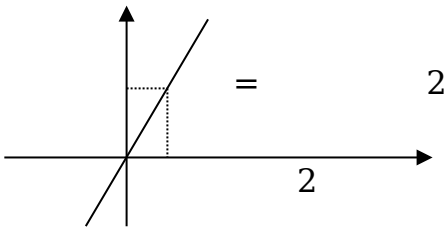
Example 4.28: Let f(x) = 2x and c = 1

Then $\lim_{x \to 1} f(x) = \lim_{x \to 1} 2x = 2$

y = 2x

and f(1) = 2(1) = 2.

Since $\lim_{x \to 1} 2x = 2 = f(1)$, f is continuous at 1.



In fact f is a continuous function. See Figure 4.4.

Figure 4.4

Remark: *For a function f to be continuous at c, the following conditions must be satisfied*

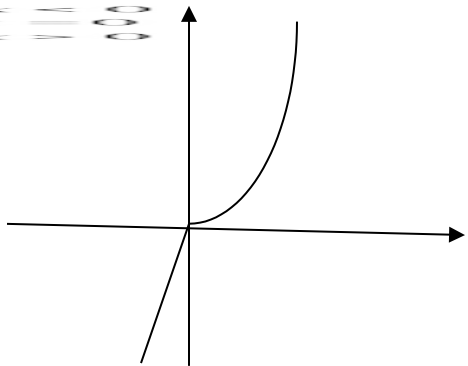
a. f(c) must be defined

b. $\lim_{x \to c} f(x)$ *must exist*

c. $\lim_{x \to c} f(x) = f(c)$

Otherwise if one of the above conditions is not satisfied, then f is discontinuous at c.

Example 4.29: Let $f(x) =$ [illegible]



Then f(0) = 2 so that f(0) is defined . $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3x = 0$ - 2

$y = x^2$

and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$. Thus $\lim_{x\to 0} f(x) = 0$ ○

But since $\lim_{x\to 0} f(x)$ ≠ f(0) , f is not continuous at 0. . $y = 3x$

Figure 4.5

Example 4.30: Let *f(x) = sinx*. Then, $\lim_{x\to \pi/2} f(x) = \lim_{x\to \pi/2} \sin x = 1 = \sin \pi/2 = f(\pi/2)$

Hence *f(x) = sinx* is continuous at $\pi/2$.

In fact *f(x) = sinx* is a continuous function. Similarly, the functions *f(x) = cosx,* the **exponential function** with base a, f(x) = a^x, the **logarithmic function** with base a, f(x) = $\log_a x$, the **natural exponential** function f(x) = e^x and the **natural logarithmic** function f(x) = nx are all continuous functions in their respective domains.

Theorem 4.6: Suppose f and g are functions with common domain such that both f and g are continuous at c. Then

1) f + g is continuous at c.
2) f – g is continuous at c.
3) if k is a scalar, kf is continuous at c.
4) fg is continuous at c.
5) if g(c) ≠ 0, $\frac{f}{g}$ is continuous at c.

Example 4.31: Let P(x) = $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ be any polynomial of degree n, and let c∈**R**, arbitrary. Then,

$\lim_{x\to c} P(x) = \lim_{x\to c}(a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0) = a_nc^n + a_{n-1}c^{n-1} + \ldots + a_2c^2 + a_1c + a_0 = P(c)$

Hence, P(x) is continuous at c, and since c was taken arbitrarily, every polynomial function is continuous.

Example 4.32: Let f(x) = $\frac{p(x)}{q(x)}$ be any rational function. Then if c is any real number such that q(c) ≠ 0, then $\lim_{x\to c} f(x) = \lim_{x\to c} \frac{p(x)}{q(x)} = \frac{\lim_{x\to c} p(x)}{\lim_{x\to c} q(x)} = \frac{p(c)}{q(c)} = f(c)$

Thus any rational function is continuous in its domain.

From the above theorem we can see that $f(x) = 5x^2 - 4x + 7$ is continuous in **R,** $g(x) = \frac{x^3 + x - 1}{x^2 - 4}$ is continuous in **R**\{-2, 2}, $h(x) = |x| \cos x - \frac{3}{x^2}$ is continuous for $x \neq 0$ and $f(x) = \frac{5}{x - 1} + \ell nx$ is continuous for $x \in (0, 1) \cup (1, \infty)$.

As a generalization of the Power Rule for limits, we have the following theorem

Theorem 4.7 (**Substitution Rule**): Suppose f and g are real valued functions such that $\lim_{x \to c} f(x) = L$ and g is continuous at L. Then $\lim_{x \to c} g(f(x)) = g\left(\lim_{x \to c} f(x)\right) = g(L)$

Example 4.33: For $f(x) = \sin x$, $g(x) = \sqrt{x}$, and $c = \pi/2$, we have $\lim_{x \to \pi/2} f(x) = 1$ and g is continuous at 1. Thus $\lim_{x \to c} g(f(x)) = \lim_{x \to c} \sqrt{\sin x} = \sqrt{1} = 1$.

Using Substitution Rule we have continuity of the composite of two functions as given by the following theorem.

Theorem 4.8: Suppose f and g are functions such that f is continuous at c and g is continuous at f(c). Then, gof is continuous at c.

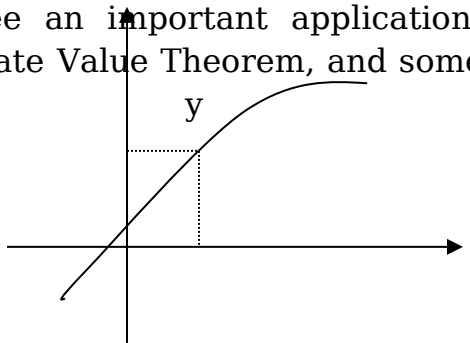
Proof: Since f is continuous at c, $\lim_{x \to c} f(x) = f(c)$. Now $\lim_{x \to c} (gof)(x) = \lim_{x \to c} g(f(x)) = g\left(\lim_{x \to c} f(x)\right) = g(f(c)) = (gof)(c)$. Therefore, gof is continuous at c.

Example 4.34: For $f(x) = x^2 + 5$, $g(x) = e^x$ and $c = 1$, we have $(gof)(x) = g(f(x)) = e^{x^2+5}$ and

$\lim_{x \to 1} (gof)(x) = \lim_{x \to 1} e^{x^2+5} = e^6 = e^{1^2+5} = e^6$. Thus, $(gof)(x) = e^{x^2+5}$ is continuous at 1.

- **Intermediate Value Theorem**

Recall that for a function f continuous on a closed interval [a, b] its graph can be traced between the points (a, f(a)) and (b, f(b)) without any break or interruption. In this section we shall see an important application of continuous functions: namely, the Intermediate Value Theorem, and some of its consequences.



For a function continuous on [a, b], the intermediate value property asserts that if L is any number between (intermediate to) f(a) and f(b), then there is at least one number c between a and b whose image under f is L. See Figure 4.6.

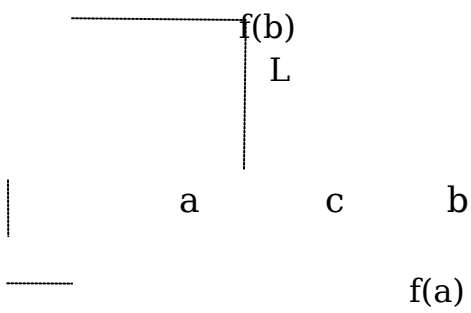


Figure 4.6

Theorem 4.9: (Intermediate Value Theorem)
Suppose f is continuous on a closed interval [a, b]. Let L be any number between f(a) and f(b), (either f(a) ≤ L ≤ f(b), or f(b) ≤ L ≤ f(a). Then there exists a number c in [a, b] such that f(c) = L.

Example 4.35: Let $f(x) = x^2$. Then f is continuous on [0, 3] with f(0) = 0 and f(3) = 9. By the Intermediate Value Theorem f assumes (takes on) every value between 0 and 9. For instance for L = 4, we have 2 ∈[0, 3] with f(2) = 4, and for L = 7, we have $\sqrt{7}$ ∈[0, 3] with $f(\sqrt{7}) = 7$.

Example 4.36: Let $f(x) = x^3 + 2x^2 + x = 4$ on [-2, 1]. Show that there exists some c ∈[-2, 1] such that f(c) = 4.

Solution: f is continuous on [-2, 1] with f(-2) = 2 and f(10 = 8. Since 2 ≤ 4 ≤ 8, it follows, by the Intermediate Value Theorem that there exists c ∈[-2, 1] such that f(c) = 4. i.e. $f(c) = c^3 + 2c^2 + c + 4 = 4$. In this case we can find such c by solving

$c^3 + 2c^2 + c + 4 = 4$
⇔ $c^3 + 2c^2 + c = 0$
⇔ $c(c^2 + 2c + 1) = 0$
⇔ $c (c + 1)^2 = 0$ which gives either c = 0 or c = -1

Since both of these values are in [-2, 1], for this particular case we have two values in [-2, 1] with image under f equal to 4.

One of the most important applications of the Intermediate Value Theorem is given in the following theorem.

Theorem 4.10: Suppose f is continuous on a closed interval [a, b] and assume that f(a) and f(b) have opposite signs. Then there is at least one c ∈(a, b) such that f(c) = 0.

Proof: Without loss of generality, assume that f(a) < 0 and f(b) > 0. Then choose L = 0, between f(a) and f(b). By the Intermediate Value Theorem, there is at least one c between a and b such that f(c) = L = 0.

Remark: *This means that the equation f(x) = 0 has at least one root in the interval (a, b).*

Example 4.37: The function $f(x) = x^3 - x - 2$ is continuous on [1, 2]. $f(1) = -2 < 0$ and $f(2) = 4 > 0$. Thus there is a number c in (1, 2) such that f(c) = 0 or $c^3 - c - 2 = 0$.

Example 4.38: Show that the graphs of $y = e^x$ and y = 3x intersect in the interval [0, 1]

Solution: Define the function $f(x) = e^x - 3x$. Then f is continuous on [0, 1] with $f(0) = e^0 - 3(0) = 1 - 0 = 1 > 0$ and $f(1) = e^1 - 3(1) = e - 3 < 0$. Thus there is a number $c \in (0, 1)$ such that $f(c) = e^c - 3c = 0$ and the graphs of $y = e^x$ and y = 3x intersect at $c \in (0, 1)$.

Exercise 4.1

1. Evaluate the following limits, if they exist.

   a. $\lim_{x\to 4}(7-2x)$ b. $\lim_{x\to 2}\frac{2x-1}{3x+1}$ c. $\lim_{x\to 3}\frac{\sqrt{x+1}}{2x-1}$

   d. $\lim_{x\to -1}\frac{x^2-1}{x+1}$ e. $\lim_{x\to 2}\frac{x-2}{\sqrt{x+2}-2}$ f. $\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}$

2. Find $\lim_{x\to c^-} f(x)$, $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c} f(x)$, if it exists, for

   a. f(x) = cos x , at $c = \pi/6$

   b. $f(x) = \sqrt{x+3}$, at c = -3

   c. $f(x) = \begin{cases} x, & for\ x < 2 \\ 2x-2, & for\ x > 2 \end{cases}$, at c = 2

   d. $f(x) = \begin{cases} |x+1|, & for\ x \geq -2 \\ 1, & for\ x < 2 \end{cases}$, at c = -2

3. Evaluate each of the following limits, if it exists.

   a. $\lim_{x\to 3}(2x^2 - 3x + 5)$ b. $\lim_{x\to 0} 2^x \sin x$ c. $\lim_{x\to \pi/4}(\cos x)^4$

   d. $\lim_{x\to 26}\sqrt[3]{\frac{x^2-4x+3}{x-3}}$ e. $\lim_{x\to 4}\left(x^{1/2} - x^{3/2}\right)$ f. $\lim_{x\to 1}\sqrt[4]{7x^3+9}$

   g. $\lim_{x\to 0}\frac{\sin x}{\tan x}$ h. $\lim_{x\to 0}\frac{\sin(2x)\sin(3x)}{5x^2}$ i. $\lim_{x\to 0}\frac{\sin 3x}{\sin 4x}$

4. Evaluate the following limits, if they exist

a. $\lim_{x \to 3^-} \frac{2}{x - 3}$ b. $\lim_{x \to 1} \frac{x}{(x - 1)^2}$ c. $\lim_{x \to \infty} \frac{4}{x - 1}$

d. $\lim_{x \to -\infty} \cos x$ e. $\lim_{x \to \infty} \frac{x^3 - 1}{2x^2}$ f.

$\lim_{x \to \infty} \frac{2x^3 + 3x - 5}{5x^3 + 1}$

5. Find all the asymptotes, if any, for the following functions

a. f(x) = tanx b. f(x) = $\frac{x^2 - 9}{x + 4}$ c.f(x) = $\frac{3x^3 + 2x + 1}{(x + 1)^2}$

6. Evaluate the following limits, if they exist.

a. $\lim_{t \to \infty} \left(1 + \frac{1}{x}\right)^{x+4}$ b. $\lim_{t \to -\infty} \left(1 + \frac{4}{x+1}\right)^{x}$ c. $\lim_{t \to \infty} \left(\frac{2x+3}{2x+1}\right)^{x-1}$

7. Check whether or not the following functions are continuous at the indicated points.

a. f(x) = x^2 + 1, at c = 2 b. f(x) = |x^2 – 1|, at c = -1, 0, 1

c. f(x) = $\frac{3}{x - 2}$, at c = 2 d. f(x) = $\begin{cases} x^2, & for\ x \neq 1 \\ 3, & for\ x = 1 \end{cases}$, at c = 1

8. Show that the following equations have roots in the indicated intervals.

a) logx = 0, in $\left[\frac{1}{2}, 2\right]$ b) 2^x – 2 = 0, in [0, 2] c) cos x – x = 0, in $\left[0, \pi/2\right]$

9. Using the Intermediate Value Theorem show that the graphs of f and g intersect in the given interval.

a. f(x) = x^3 + 4x + 2 and g(x) = -1, in [-1, 0]

b. f(x) = 2sinx and g(x) = 1 – x, in [0, 2]

c. f(x) = x ℓnx and g(x) = sinx, in $\left[\frac{1}{e}, e\right]$

4.2. Derivatives

Objectives

At the end of this section you should be able to

- get acquainted with the concept of the derivatives of a function.
- evaluate the derivative of elementary functions using the definition.
- find the slope and equation of a tangent line to a curve at a given point.
- evaluate the derivatives of combinations of functions.
- find the derivatives of polynomial and rational functions.
- have a good understanding of the Chain Rule.
- apply the Chain Rule to evaluate derivatives of composite functions and algebraic functions.
- find the derivative of the logarithmic function.
- find the derivative of the exponential function.
- apply the above derivatives to the natural logarithmic and natural exponential functions as special cases.
- evaluate derivatives of composite functions with the logarithmic and exponential functions.
- have an understanding of the derivative of a derivative.

Using the concepts discussed in section 4.1, we are now ready to study one of the central concepts of calculus: the derivative of a function. Even though the derivative is connected with finding the tangent lines to curves at a point, its main applications are in finding rates of change of variable quantities relative to the change in another quantity.

Consider a function f continuous at a point c in its domain.

Then, by definition of continuity $\lim_{x \to c} f(x)$ = f(c)

This means for x close to c, f(x) is close to f(c). If we denote the **increment** (or change) x – c in the x-direction by h = x – c (so that x = c + h) as is seen in Figure 4.7,

then the corresponding change in the y-direction

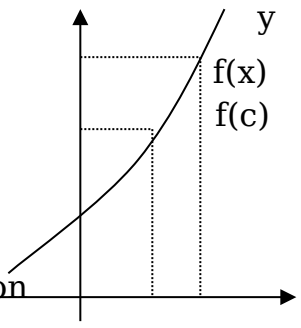


Figure 4.7

x

is given by

f(x) – f(c) = f(c + h) – f(c).

The ratio of these two increments is given by

$$\frac{f(x) - f(c)}{x - c}$$

and is called the **difference quotient of f at c.**

For instance, if $f(x) = x^2 + 2$ and c = 3, then

$$\frac{f(x) - f(3)}{x - 3} = \frac{(x^2 + 2) - (3^2 + 2)}{x - 3} = \frac{x^2 - 9}{x - 3}$$

We shall define the derivative of a function of f at c as the limit of the above difference quotient, if the limit exists.

Definition 4.11

Let c be a number in the domain of a function f. If

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists, we call this limit the **derivative of f at** c, and denote it by $f'(c)$, so that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

If this limit exists we say that f **has a derivative at c**, or f is **differentiable at c** or **$f'(c)$ exists.**

Remarks: 1. *Observe that we can alternatively write*

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

since for $h = x - c$, we have $x = c + h$ and as $x \to c$, $h \to 0$.

2. *The notation $f'(c)$ is read as "the derivative of f at c" or for short "f prime at c".*

Other notations are given by $\frac{df}{dx}(c)$ *or Df(c)*

3. *The quantity $f'(c)$ describes the rate of change of the function f around the point (c, f(c)).*

Example 4.39: Let f(x) = 2x + 3. Then, for any c ∈**R**, the point (c, f(c)), we have

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{(2x+3) - (2c+3)}{x - c} = \lim_{x \to c} \frac{2x - 2c}{x - c} = 2 \lim_{x \to c} \frac{x - c}{x - c} = 2 \lim_{x \to c}$$

(1) = 2. Since c∈**R** is arbitrarily taken, we have for f(x) = 2x + 3, $f'(x) = 2$ for all x∈**R.**

In fact for any linear function f(x) = ax + b, we have

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{(ax+b) - (ac+b)}{x - c} = a \lim_{x \to c} \frac{x - c}{x - c} = a$$

for any c ∈**R**. Thus $f'(x) = a$

Note that the graph of a linear function is a straight line and the rate of change (a constant) is measured by the slope of the line.

Example 4.40: Let $f(x) = 3x^2 + 5$. Then for any $x \in \mathbf{R}$

$f(x + h) = 3(x + h)^2 + 5 = 3x^2 + 6xh + 3h^2 + 5$ and

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(3x^2 + 6xh + 3h^2 + 5) - (3x^2 + 5)}{h}$$

$$= \lim_{h\to 0}\frac{6xh + 3h^2}{h} = \lim_{h\to 0}\frac{3h(2x + h)}{h} = 3\lim_{h\to 0}(2x + h) = 6x.$$

Thus, for $f(x) = 3x^2 + 5$, $f'(x) = 6x$ for any $x \in \mathbf{R}$.

In particular, when $c = 1$, $f'(1) = 6(1) =. 6$ is the slope of the tangent line to the graph of f at (1,8)

Example 4.41: Let $f(x) = c$, where c is a constant.

Then for any $x \in \mathbf{R}$,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$

Thus, for $f(x) = c$, a constant, $f'(x) = 0$ for all $x \in \mathbf{R}$.

Hence, for $f(x) = 15$, $f'(x) = 0$, for $f(x) = -\sqrt{2}$, $f'(x) = 0$, and so on.

Applying the above definition, we can get the following derivatives.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$f(x) = \frac{1}{x}$	$f'(x) = \frac{-1}{x^2}$, for all $x \neq 0$	$f(x) = \sin x$	$f'(x) = \cos x$, for all $x \in \mathbf{R}$
$f(x) = \sqrt{x}$	$f'(x) = \frac{1}{2\sqrt{x}}$, for $x > 0$	$f(x) = \cos x$	$f'(x) = -\sin x$, for all $x \in \mathbf{R}$,

Using the definition to evaluate the derivative of more complicated combinations and compositions of functions becomes cumbersome. At this stage the student must be able how to find the derivatives of various types of functions quickly and efficiently without always resorting to the definition. In the table below we list some techniques of differentiation which can be p

Theorem 4.11: Suppose f and g are differentiable at c, and k is a constant, then

a) $(kf)'(c) = k f'(c)$ … **Constant Rule**

b) $(f + g)'(c) = f'(c) + g'(c)$ … **Addition Rule**

c) $(f - g)'(c) = f'(c) - g'(c)$ … **Difference Rule**

d) $f(x) = x^n$, n an integer, $f'(x) = n x^{n-1}$ … **Power Rule**

d) $(fg)'(c) = f'(c)\, g(c) + f(c)\, g'(c)$ … **Product Rule**

Thus, if $f(x) = x^4$, then $f'(x) = 4x^3$ and if $g(x) = x^{12}$, then $f'(x) = 12x^{11}$, and so on.

Example 4.42: Let $f(x) = x^2 + 3$ and $g(x) = \sin x$. Then

$(f + g)'(x) = f'(x) + g'(x) = \frac{d}{dx}(x^2 + 3) + \frac{d}{dx}(\sin x) = 2x + 0 + \cos x = 2x + \cos x$

$\frac{d}{dx}(g(x) - 4f(x)) = \frac{d}{dx}(\sin x) - 4\frac{d}{dx}(x^2 + 3) = \cos x - 4(2x + 0) = \cos x - 8x$.

Since polynomials are sums or differences of constant multiples of powers of x, the first four rules help us to evaluate their derivatives.

Remark: *Given a polynomial of degree n,* $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$

$P'(x) = \frac{d}{dx}(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0)$

$= a_n\frac{d}{dx}(x^n) + a_{n-1}\frac{d}{dx}(x^{n-1}) + \ldots + a_2\frac{d}{dx}(x^2) + a_1\frac{d}{dx}(x) + \frac{d}{dx}(a_0)$.

$= na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2x + a_1$

Example 4.43: For $p(x) = 5x^4 - 2x^3 + x^2 + 7x - 1$, we have $p'(x) = 20x^3 - 6x^2 + 2x + 7$.

For $q(x) = 6x^3 + \sqrt{2}x^2 - 3x + \pi$, we have $q'(x) = 18x^2 + 2\sqrt{2}x - 3$.

As an application of the product rule, we have the following examples.

Example 4.44: Let $k(x) = 2x \sin x$. Find $k'(x)$.

Solution: If we put $f(x) = 2x$ and $g(x) = \sin x$, then $f'(x) = 2$ and $g'(x) = \cos x$.

Thus, $k'(x) = f'(x)g(x) + f(x)g'(x) = 2\sin x + 2x\cos x$.

Remark: *In practice, to evaluate the derivative of a product of two functions, we do not need to identify which one is f and which one is g.*

Example 4.45: Let $h(x) = x^3 \cos x$. Then

$h'(x) = (x^3)' \cos x + x^3(\cos x)' = 3x^2\cos x + x^3(-\sin x) = 3x^2\cos x - x^3 \sin x$.

For the derivative of the product of three functions f, g and h, we have

$(fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

Example 4.46: Let $k(x) = x^3 \sin x \cos x$. Find $k'(x)$.

Solution: Put $f(x) = x^3$, $g(x) = \sin x$ and $h(x) = \cos x$ in the above statement with $f'(x) = 3x^2$, $g'(x) = \cos x$ and $h'(x) = -\sin x$. Then $k'(x) = 3x^2 \sin x \cos x + x^3 \cos x.\cos x + x^3\sin x(-\sin x) = 3x^2 \sin x \cos x + x^2 \cos^2 x - x^3 \sin^2 x$.

The Quotient Rule is used to find the derivative of any rational function. If $f(x) = \frac{p(x)}{q(x)}$, *for p, q polynomials, we then have* $f'(x) = \left(\frac{p(x)}{q(x)}\right)^1 = \frac{p'(x)q(x) - p(x)q'(x)}{(q(x))^2}$, *for* $q(x) \neq 0$.

Example 4.47: Let $f(x) = \frac{3x^2 - 5}{2x+1}$. *Find* $f'(x)$

Solution: Putting $p(x) = 3x^2 - 5$ and $q(x) = 2x + 1$, we get

$$f'(x) = \frac{6x(2x+1) - (3x^2 - 5)(2)}{(2x+1)^2} = \frac{12x^2 + 6x - 6x^2 + 10}{(2x+1)^2} = \frac{6x^2 + 6x + 10}{(2x+1)^2}$$

As an important consequence of the Quotient Rule, we can now find the derivatives of the remaining four trigonometric functions.

Example 4.48: Let $f(x) = \tan x$. Show that $f'(x) = \sec^2 x$

Solution: $f(x) = \tan x = \frac{\sin x}{\cos x}$. Then

$$f'(x) = \frac{d}{dx}(\tan x) = \left(\frac{\sin x}{\cos x}\right)^1 = \frac{(\sin x)^1 \cos x - \sin x(\cos x)^1}{(\cos x)^2}$$

$$= \frac{\cos x.\cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

In the same manner, we can show that

$$\frac{d}{dx}(\cot x) = -\csc^2 x, \quad \frac{d}{dx}(\sec x) = \sec x \tan x \quad \text{and} \quad \frac{d}{dx}(\csc x) = -\csc x \cot x.$$

The Chain Rule states that $(gof)'(x) = g'(f(x))\, f'(x)$, for all x such that f is differentiable at x and g is differentiable at f(x).

Example 4.49: Find the derivative of $h(x) = \cos(x^2 + 1)$

Solution: Let $f(x) = x^2 + 1$ and $g(x) = \cos x$. Then, $h(x) = (gof)(x) = g(f(x)) = g(x^2 + 1) = \cos(x^2 + 1)$ and $h'(x) = g'(f(x))$. $f(x) = -\sin(x^2 + 1) \,.\, (x^2 + 1)' = 2x\sin(x^2 + 1)$.

If a and b are any real numbers, we can easily show that

$$\frac{d}{dx}(\sin ax) = a\cos ax \quad \text{and} \quad \frac{d}{dx}(\cos bx) = -b\sin bx$$

Thus, $\frac{d}{dx}(\sin 4x) = 4\cos 4x$ and $\frac{d}{dx}(\cos 5x) = -5\sin 5x$

Example 4.50: Find the derivative of $h(x) = (1+3x-5x)^{12}$

Solution: Let $f(x) = 1 + 3x - x^5$ and $g(x) = x^{12}$. Then h = gof and

$h'(x) = \frac{d}{dx}(1+3x - x^5)^{12} = 12(1 + 3x - x^5)^{11}\,(1 + 3x - x^5)' = 12(3 - 5x^4)\,(1 + bx - x^5)^{11}$.

Example 4.51: Find the equations of tangent and normal lines to the semicircle

$$y = f(x) = \sqrt{1- x^2} \text{ at } \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

Solution: The slope of the tangent line T is given by the derivative of $y = f(x) = \sqrt{1- x^2}$ at $x = \frac{1}{2}$. Thus, by Chain Rule,

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}\sqrt{1- x^2} = \frac{1}{2\sqrt{1- x^2}}(1- x^2)' = \frac{- 2x}{2\sqrt{1- x^2}} = \frac{- x}{\sqrt{1- x^2}}$$

so that the slope of T is

$$m = f'\left(\frac{1}{2}\right) = \frac{- 1/2}{\sqrt{1- 1/4}} = -\frac{1}{2}\cdot\frac{2}{\sqrt{3}} = \frac{- 1}{\sqrt{3}}$$

and since the tangent line passes through the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, its equation in slope-point form is

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}\left(x - \frac{1}{2}\right) \quad \text{or} \quad x + \sqrt{3}y - 2 = 0$$

The slope of the normal line at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is $\sqrt{3}$ and its equation is

$$y - \frac{\sqrt{3}}{2} = \sqrt{3}\left(x - \frac{1}{2}\right) \quad \text{or} \quad y - \sqrt{3}x = 0.$$

Remark: *The Chain Rule can be extended to more than two functions. Suppose k(x) = (hogof)(x) = h(g(f(x))) and let f be differentiable at x, g be differentiable at f(x) and h be differentiable at g(f(x)). Then k′(x) = (hogof)′(x) = h′(g(f(x)).g′f(x)). f′(x)*

Similarly, if ℓ(x) = (kohogof)(x) = k(h(g(f(x)))), then

$\ell'(x) = (kogohof)'(x) = k'(h(g(f(x)))).\ h'(g(f(x)).g'(f(x)).f'(x).$

You can now see why this method is called the Chain Rule!

Example 4.52: Find the derivative of the function

$$k(x) = \cos\sqrt{2x^2 - 3}$$

Solution: Let $k(x) = \cos\sqrt{2x^2 - 3} = h(g(f(x)))$ with

$f(x) = 2x^2 - 3$, $g(x) = \sqrt{x}$ and $h(x) = \cos x$. Then

$$k'(x) = \frac{d}{dx}\left(\cos\sqrt{2x^2 - 3}\right) = h'(g(f(x)).\ g'(f(x))\ f'(x)$$

$$= -\sin\sqrt{2x^2 - 3}\ .\ \frac{1}{2\sqrt{2x^2 - 3}}.4x\ =\ \frac{-\ 2x\sin\sqrt{2x^2 - 3}}{\sqrt{2x^2 - 3}}$$

Example 4.53: Let f(x) = sin(tanx^2). Find $f'(x)$

Solution: $f'(x) = \frac{d}{dx}(\sin(\tan x^2))$

$= \cos(\tan x^2)\sec^2 x^2(2x) = 2x.\cos(\tan x^2)\sec^2 x^2.$

- **Derivatives of Logarithmic and Exponential Function**

Recall that for $a > 0$, and $a \neq 1$, the logarithmic function with base a is given by

$$f(x) = \log_a x \qquad \text{for } x > 0.$$

In particular, when a = e, we get the natural logarithmic function

$$f(x) = \log_a x = \ell nx \quad , \quad \text{for } x > 0.$$

Theorem 4.12: Let a > 0, a ≠ 1 and let f(x) = $\log_a x$. Then

$$f'(x) = \frac{1}{x}\log_a x$$

From Theorem 4.12, when the base a = e , it follows that

$$(\log_e x)' = (\ell nx)' = \frac{1}{x}\log_e e = \frac{1}{x}.1 = \frac{1}{x} : \boxed{(\ln x)' = \frac{1}{x}}$$

Also, by applying change of base of logarithms, we get

$$(\log_a x)' = \frac{1}{x}\log_a e = \frac{1}{x}\frac{\log_e e}{\log_e a} = \frac{1}{x}.\frac{1}{\ln a} : \text{i.}\ \boxed{(\log_a x)' = \frac{1}{x\ln a}}$$

Example 4.54: For $f(x) = \log_5 x$, we have $f'(x) = \frac{1}{x}\log_5 e = \frac{1}{x\ln 5}$

Example 4.55: Find the derivative of the following

a) f(x) = $\log_3(x^2 + x - 1)$ b) g(x) = $\frac{x}{\ln x}$

Solution: a) $f'(x) = \frac{1}{x^2 + x - 1}.\log_3 e\ (x^2 + x - 1)' = \frac{2x+1}{x^2 + x - 1}.\log_3 e$

b) g□(x) = $\left(\frac{x}{\ln x}\right)' = \frac{1.\ln x - x.\frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{\ln^2 x}$

Theorem 4.13: Let a > 0, a ≠ 1 and let f(x) = a^x. Then, $f'(x) = (a^x)' = \frac{a^x}{\log_a e}$.

By applying change of base we also have

$$(a^x)' = \frac{a^x}{\log_a e} = a^x \ell n$$

When the base a = e, we get $(e^x)' = \frac{e^x}{\log_e e} = e^x \ell ne = x$ $(e^x)' = e^x$ *e.*

Example 4.56: For f(x) = 3^x, we have $f'(x) = \frac{3^x}{\log_3 e} = 3^x \ell n3$.

Example 4.57: Find the derivative of the following

a) f(x) = $e^{\sqrt{x+1}}$ b) g(x) = $3^{\sin x}$

c) f(x) = $\sqrt{x+e^{4x}}$ d) g(x) = $e^{x^2} \ell nx$

Solution: a) By using the Chain Rule, we get

$$f'(x) = \left(e^{\sqrt{x+1}}\right)' = e^{\sqrt{x+1}}.\left(\sqrt{x+1}\right)' = e^{\sqrt{x+1}}, \frac{1}{\sqrt{x+1}} = \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}$$

b) $(3^{\sin x})' = 3^{\sin x}, \ell n3.(\sin x)' = \ell n3.\cos x.3^{\sin x}$

c) $\frac{d}{dx}\sqrt{x+e^{4x}} = \frac{1}{2\sqrt{x+e^{4x}}} (x + e^{4x})' = \frac{1+4e^{4x}}{2\sqrt{x+e^{4x}}}$

d) By the Product Rule and Chain Rule we get:

$$g'(x) = \left(e^{x^2} \ln x\right)' = 2xe^{x^2} \ell nx + e^{x^2}.\frac{1}{x} = e^{x^2}\left(2x\ln x + \frac{1}{x}\right)$$

- **Higher Derivatives**

If a function f is differentiable at a point x in its domain, we denote its derivative by $f'(x)$, where

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$, provided the limit exists.

This derivative is usually called the **first derivative of f at x.**

If the new function f' is differentiable at a point x, then we can repeat the process and find its derivative as

$$(f'(x))' = f''(x) = \lim_{h\to 0}\frac{f'(x+h) - f'(x)}{h},$$ provided the limit exists.

we call $f''(x)$ the **second derivative of f at *x***, and it is often read as "f double prime of x".

Observe that $f''(x)$ is simply the derivative of the function f' at x and is no more difficult than finding the first derivative.

Example 4.58: If $f(x) = \ell nx$, then $f'(x) = \frac{1}{x}$ and hence $f''(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$.

We can similarly find the derivative of $f''(x)$ to get

$(f''(x))' = f'''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h}$, and so on,

and call this the **third derivative of f at x**.

Thus, for $f(x) = \ell nx$, $f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$ and $f'''(x) = \frac{2}{x^3}$

*These derivatives when they exist are called **higher derivatives (or derivatives of derivatives)***

The n^{th} derivative $f^{[n]}(x)$ can also be denoted by $\frac{d^n f}{dx^n}(x)$.

Thus the second derivative is $f''(x)$ or $f^{[2]}(x)$ or $\frac{d^2 f}{dx^2}(x)$ and the third derivative is $f'''(x)$ or $f^{[3]}(x)$ or $\frac{d^3 f}{dx^3}(x)$.

Example 4.59: Find the higher derivatives of the following

a) $f(x) = 4x^3 + x^2 - 3x + 7$ b) $g(x) = e^x$

c) $f(x) = \sin x$ d) $g(x) = \ell n(3x)$

Solution: a) For $f(x) = 4x^3 + x^2 - 3x + 7$, we have

$f'(x) = 12x^2 + 2x - 3$

$f''(x) = 24x + 2$

$f'''(x) = 24$

$f^{[4]}(x) = 0$ and for $n \geq 4$, $f^{[n]}(x) = 0$

b) For $g(x) = e^x$, $g'(x) = e^x$, $g''(x) = e^x$, and in general for $n > 1$, $g^{[n]}(x) = e^x$

c) $f(x) = \sin x$, $f'(x) = \cos x$

$f''(x) = -\sin x$, $f'''(x) = -\cos x$

$f^{[4]}(x) = \sin x$ and so on

d) $g(x) = \ell n(3x)$, $g'(x) = \frac{3}{3x} = \frac{1}{x}$ by Chain Rule

$g''(x) = -\frac{1}{x^2}$, $g'''(x) = \frac{2}{x^3}$, $g^{[4]}(x) = -\frac{3.2}{x^4}$, ...

Exercise 4.2

1. For each of the following functions, find $f'(c)$ using the definition
   a. f(x) = 2x – 4, at c = 1 b. f(x) = x^2 + 3, at c = -1
   c. f(x) = x^3 – 2 at c = 0 d. f(x) = |x + 2|, at c = 2
2. Find the equations of the tangent and normal lines to the graph of f at the given point.
   a. f(x) = x^2 + x – 1, at (2, 5) b. f(x) = $\sqrt{x}$, at (4, 2)
   c. f(x) = 2cosx, at ($\pi/2$, 0) d. f(x) = $\frac{1}{x}$, at (2, $\frac{1}{2}$)
3. Find the derivative of the following functions
   a. f(x) = (x^2 – 5) cosx b. g(x) = $\sqrt{x}$ secx
   c. $\frac{2x^3 - 5x}{x^2 + 3}$ d. g(x) = $\frac{2x}{\tan x}$
4. Find the equations of the tangent and normal lines to the functions at the indicated point.
   a. f(x) = sinx cosx, at $(\pi/4, 1/2)$
   b. f(x) $\frac{x}{x^2 + 1}$, at $(1, 1/2)$
5. Find the derivative of the following functions.
   a. f(x) = $\tan^3 x$ b. g(x) = x$\sqrt{1 - x^2}$ c. f(x) = $\sqrt{x}$ sin x^2
   d. g(x) = $\frac{1}{(x^2 - x)^4}$ e. f(x) = $\sqrt{x^2 + \sqrt{x^2 + 1}}$ f. g(x) = xcosx + $\sqrt[3]{5x - 4}$
   g. f(x) = sin$\sqrt{2x + 1}$ h. g(x) = $\frac{\cot x}{\sqrt{x + 1/x}}$ i. f(x) = $e^{\tan x}$
   j. g(x) = ℓn(ℓnx) k. f(x) = (ℓnx + $e^{\sqrt{x}}$)^3 l. f(x) = $\ell n^2 x$ + ℓnx^2
6. Find the first, second and third derivatives of the following function
   a. f(x) = e^{x^2} b. g(x) = secx
   c. f(x) = sin(2x) + cos(3x) d. g(x) = ℓn(sinx)

4.3. Applications of the derivative

At the end of this section you should be able to:

- define maximum and minimum values of a function on a given interval.

- explain the fundamental theorem of local extrema values.
- identify the regions where a function is increasing and decreasing.
- apply the first and second derivative tests to find local extrema values of a function.
- solve practical problems related to extrema.
- state the important points that are necessary to sketch the graph of a function.
- sketch the graph of a function applying the above concepts.
- solve related rates problems.

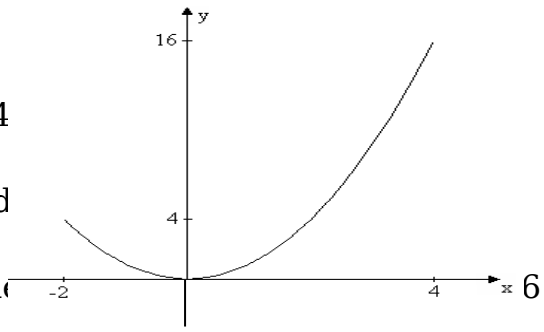
At the beginning of this unit we have mentioned that the derivative of a function at a point c in its domain measures the rate of change of the function around that point. In this section we shall see how the derivative can be applied to solve a variety of problem in the areas of engineering, the natural sciences, business and the social sciences. We see how it can be used to solve maximum and minimum values of a function (i.e., where it has "peaks" and where it has "valleys"), where it curves upward and where it curves downward, and in general, to sketch the graph of the function. At the end we shall introduce related rates problems and see how to solve them using the derivative.

a) Extrema of a Function

Definition. Let f be a function defined on an interval I. If there is a number d in I such that $f(x) \leq f(d)$ for all x in I, then f(d) is called the **maximum value of f on I.** Similarly, if there is a number c in I such that $f(x) \geq f(c)$ for all x in I, then f(c) is called the **minimum value of f on I**. (See Figure 4.8) A value of f that is either a maximum value or a minimum value of f on I is called an **extreme value of f on I**.

Remark: *If the set I is the domain of the function f and if f has a maximum value on I, then this maximum value is called the (**absolute or global) maximum** of f.*
Similar for minimum value of f.

Example 4.60: Let $f(x) = x^2$ on I = [-2, 4



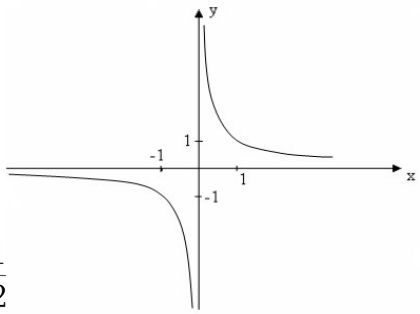
f has the maximum value of 16 = f(4) and the minimum value of 0 = f(0). Both 0 and 16 are extreme values of f.
- On the interval [-2, 4), the minimum valu
f is 0 but f has no maximum.

- On the interval (0, 4) f has neither a maximum nor a minimum. See Figure 4.8. Figure 4.8

Example 4.61: Let f(x) = $\frac{1}{x}$ for x ≠ 0.

The domain of f is I = (-∞, 0) ∪ (0, ∞) and f has neither a maximum nor a minimum value on I. See Figure 4,10

- On the interval [-1, 0] f has the maximum value -1 = f(-1), but no minimum.
- On the interval (0, 2] f has the minimum value $\frac{1}{2}$ but has no maximum.
- On the interval [-1, 2], f has no extrema. Figure 4.9



Graph of $f(x) = \frac{1}{x}$

Note that in the first example when the interval is open we have no extrema, while in the second example, when the function is not continuous, we had no extrema. Continuity of a function on a closed interval gives us a sufficient condition for the existence of both extreme values.

> **Theorem 4.14:** (Maximum-Minimum Theorem). Let f be continuous on a closed bounded interval [a, b]. Then f has a

Hence the function f(x) = x^2 for -2 ≤ x ≤ 4 has both extreme values on [-2, 4]. Similarly, the function f(x) = x^3 – 4x + 5 for 0 ≤ x ≤ 2 which is continuous on [0, 2] has a maximum and a minimum value on [0, 2], by Theorem 4.14. Even though the above theorem tells us about the existence of extreme values on [a, b], it does not tell us where they occur or how to find them. The following theorem will help us in determining such values.

> **Theorem 4.15:** Let f be defined and continuous on [a, b]. If f has an extreme value at c in (a, b) and f is differentiable at c, then

Example 4.62: Let $f(x) = x^3 - 3x + 1$.
Then f is differentiable and the critical points of f are the values of x for which $f'(x) = 0$,

But $f'(x) = 3x^2 - 3 = 0$

$\Leftrightarrow 3(x - 1)(x + 1) = 0$

$\Rightarrow$ x = 1 and x = -1 are critical points of f.

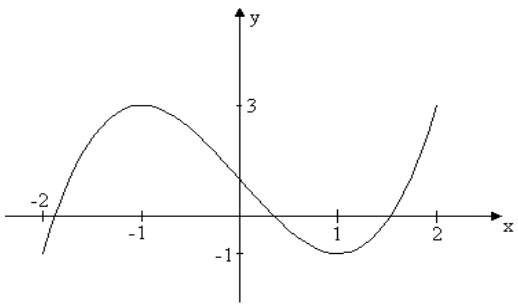


Figure 4.10

If we want to find extreme values of f on, say, the interval [-3, 3] we compute and compare the values of f at -3, -1, 1 and 3 to get f(-3) = 17 , f(-1) = 3, f(1) = -1 and f(3) = 19.

Thus the minimum value of f on [-3, 3] is -17 which occurs at -3 and the maximum value of f is 19 which occurs at 3.

- **Monotonic Functions**

One of the important points needed to sketch the graph of a function is to find the regions in which the graph slopes upward to the right (increases) or it slopes downward to the right (decreases) as seen in Figure 4.11 (a) and (b), respectively.

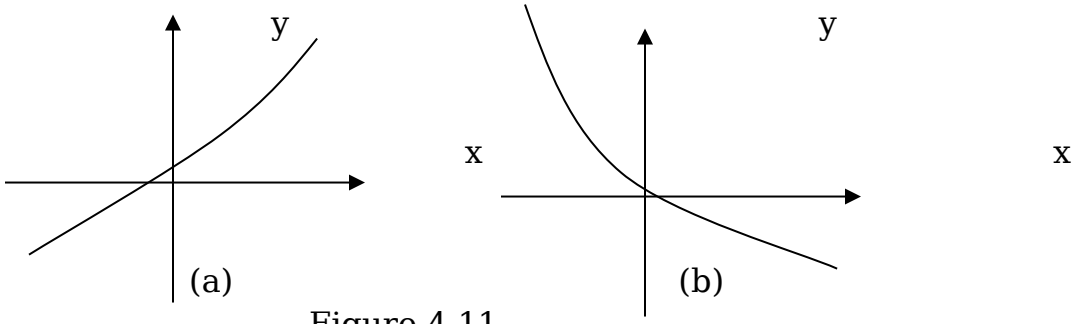


Figure 4.11

Definition 4.12: Suppose f is a function defined on an interval I.

i) f is said to be **increasing** on I if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$

ii) f is said to be **decreasing** on I if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$

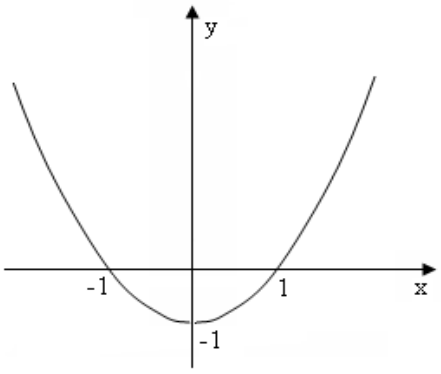
iii) f is said to be **monotonic** on I if f is either increasing or decreasing on I.

Remark: *we can similarly define the terms* ***strictly increasing, strictly decreasing*** *and* ***strictly monotonic*** *by replacing* $\leq$ *by* $<$ *and* $\geq$ *by* $>$.

Example 4.63: Let $f(x) = x^2-1$.
Find the intervals of monotonicity of f.

Solution: For $x_1, x_2 \in (-\infty, 0)$ with $x_1 < x_2$, we have

$f(x_1) = x_1^2 - 1 > x_2^2 - 1 = f(x_2)$



⇒ f is strictly decreasing on $(-\infty, 0)$.

For $x_1, x_2 \in (0, \infty)$ with $x_1 < x_2$, we have

$f(x_1) = x_1^2 - 1 < x_2^2 - 1 = f(x_2)$

⇒ f is strictly increasing on $(0, \infty)$. Figure 4. 12

The derivative of a function gives us a test for monotonicity as is indicated in the following theorem.

Theorem: 4.16 Suppose f is continuous and differentiable on an interval I.

i) If $f'(x) > 0$, for every $x \in I$, then f is strictly increasing

Example 4.64: Find the intervals over which the following function $f(x) = x^3 - 3x + 1$is monotonic.

Solution: For $f(x) = x^3 - 3x + 1$, $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$

To find the intervals over which f is increasing and decreasing we find the sign of $f'(x)$ using the critical points $x = 1$ and $x = -1$ and the Sign Chart Method.

		-1		1	
x – 1	- - - - - - - - - - - - - - -	0	+ + + + + ++		+ +
x + 1	- - - - - - -	0	+ ++ + + + + + + + + + + + +		
$f'(x)$	+ + + + +	0	- - - - - -	0	+ + + + + + + + + +

From the above "sign chart" we can see that

$f'(x) > 0$ for $x \in (-\infty, -1) \cup (1, \infty)$ and $f'(x) < 0$ for $x \in (-1, 1)$.

Thus f is strictly increasing on $(-\infty, -1) \cup (1, \infty)$ and strictly decreasing on [-1, 1].See Figure 4.10.

- **The First and Second Derivative Tests for Relative Extrema**

If f is a differentiable function, we have seen that at relative extreme values $f'(c) = 0$. Thus in order to locate relative extreme values of f we find the values of x for which $f'(x) = 0$ or $f'(x)$ does not exist. But this method does not help us to determine which of these values of x give relative extreme values (or which value is a maximum or which is a minimum). The next two theorems will provide us with conditions that guarantee that f has relative extr of fun

Theorem 4.17: (The First Derivative Test)

Let f be continuous on an interval I, and let $c \in I$.

a) If $f'(x)$ changes its sign from positive to negative at c
i.e. if $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c, then f has a relative maximum value at c.

b) If $f'(x)$ changes its sign from negative to positive at c, then f has a relative minimum value at c.

Example 4.65: Consider again the function f(x) = x^3 – 3x + 1.
$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1) = 0$ gives the critical points x = 1 and x = -1
For the critical point x = -1 check the sign of f' at -2 and 0 with $f'(-2) = 9 > 0$ and $f'(0) = -3 < 0$. Thus $f'(-1) = 3$ is a relative maximum value of f.
Similarly taking the critical point x = 1 between 0 and 2, we get $f'(0) = -3 < 0$ and $f'(2) = 9 > 0$. Thus f(1) = -1 is a relative minimum value of f. (See Figure 4.11 above)

The above theorem needs to check the signs of two distinct points to the left and to the right of each critical point. The next theorem makes use of the sign of the second derivative directly at the critical points.

Theorem: 4.18 (The Second Derivative Test)
Let f be differentiable in an interval I and let c ∈I with $f'(c) = 0$.
a) If $f''(c) < 0$, then $f(c)$ is a relative maximum value of f.
b) If $f''(c) > 0$, then $f(c)$ is a relative minimum value of f.
If $f''(c) = 0$, then we can not draw any conclusion about f(c).

Example 4.66: Consider again the function f(x) = x^3 – 3x + 1 with $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$. We have $f'(1) = f'(-1) = 0$ and $f''(x) = 6x$. Since $f''(-1) = -6 < 0$, $f(-1) = 3$ is a local maximum value of f. Since $f''(1) = 6 > 0$, $f(1) = -1$ is a local minimum values of f.

Example 4.67: Let f(x) = $\frac{4x}{x^2+4}$, Find the local extreme values of f.

Solution: $f'(x) = \frac{4(x^2+4) - 4x(2x)}{(x^2+4)^2}$, Quotient Rule.

$= \frac{4x^2+16-8x^2}{(x^2+4)^2} = \frac{16-4x^2}{(x^2+4)^2}$

$f'(x) = 0 \Rightarrow 16 - 4x^2 = 0 \Rightarrow x = 2$ or $x = -2$

$f''(x) = \dfrac{-8x(x^2+4)^2 - (16-4x^2)2(x^2+4)(2x)}{(x^2+4)^4}$ - Quotient Rule and Chain Rule

$= \dfrac{8x(x^2-12)}{(x^2+4)^3}$ - Simplification.

Thus $f''(2) = \dfrac{16(-8)}{8^3} = \dfrac{-1}{4} < 0$ $\Rightarrow$ f(2) = 1 is a local maximum value of f and

$f''(-2) = \dfrac{-16(-8)}{8^3} = \dfrac{1}{4} > 0$ $\Rightarrow$ f(-2) = -1 a local minimum value of f.

- **Practical Applications of the Extrema**

A lot of practical problems can be expressed as a continuous function on a closed and bounded interval we may be interested to find points where f attains its maximum or its minimum values. For instance we may be interested in finding the maximum area of a region to be enclosed by a fixed perimeter; the minimum distance from a fixed point to a curve. In economics a function may represent a profit or cost function and we may want to find the value of x to find maximum profit and minimum cost, and so on. The Maximum – Minimum Theorem and the first and second derivative test will be crucial in finding such points as are illustrated in the following examples.

Example 4.68: A landowner wishes to use 2000 meters of fencing to enclose a rectangular region. Suppose one side of the land lies along a river and does not need fencing. What should be the sides of the region in order to maximize the area?

Solution: Suppose the rectangle is to have length x and width y meters as seen in Figure 4.13.

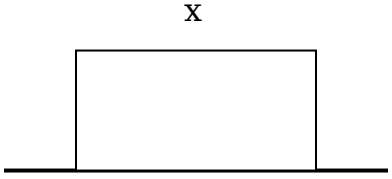


Figure 4.13

Since the length of the fencing is 2000 meters, we have

$x + 2y = 2000$

$\Rightarrow \quad 2y = 2000 - x \Rightarrow y = 1000 - x/2$

The area of the rectangle is A = xy which can be written as a function of x alone as

$$A(x) = xy = x(1000 - x/2) = 1000x - \frac{x^2}{2} \qquad \text{for } 0 \le x \le 2000$$

Thus we find the maximum value of A on [0, 2000].

$A'(x) = 1000 - x = 0 \quad \Rightarrow \quad x = 1000$ is a critical point.

Comparing the value of A at the critical point and at the endpoints 0 and 2000, we get

$A(0) = 0$, $A(1000) = 500{,}000$ and $A(2000) = 0$ (check!)

Thus the maximum value of A occurs when x = 1000 so that

$y = 1000 - x/2 = 1000 - 500 = 500$.

Consequently, to enclose maximum area, the fence should have a length of 1000 mts and a width of 500 mts.

Example 4.69: Ethiopian Airlines offers a round trip discount on group flight from Addis Ababa to Lalibela. If x people sign up for the flight, the cost of each ticket is to be 1000 – 2x Birr. Find the number of people the airline gets maximum revenue from the sales of tickets for the flight,

Solution: Since individual cost of a ticket is 1000 – 2x, the total cost of the group will be

$C(x) = (1000 - 2x)x = 1000x - 2x^2$.

To find a critical point, we solve $C'(x) = 1000 - 4x = 0$, which gives the only critical point

$x = 250$ of C(x).

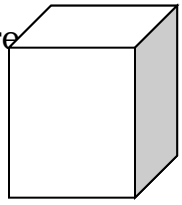
You can easily check that for $x < 250$, $C'(x) > 0$ and for $x > 250$, $C'(x) < 0$. Thus by the First Derivative Test C has an (absolute) maximum value at x = 250.

The maximum revenue the airline gets from the sales of 250 tickets is then

$C(250) = 1000(250) - 2(250)^2 = 125{,}000$ Birr.

Example 4.70: A manufacturer wishes to produce rectangular containers with square bottom and top each of which is to have a capacity of 1000 cubic inches. If the cost of production of each container is proportional to its surface area, what should be the dimensions so as to minimize the cost of production?

Solution: Let x be the side of the base and h be the height of the container as seen in Figure



Then the volume is

$$V = x^2h = 1000$$

$$\Rightarrow \quad h = \frac{V}{x^2} = \frac{1000}{x^2} \quad \text{for} \quad x > 0$$

Figure 4.15

To find the surface area, we have the area of the top and bottom as $2x^2$ and the area of the four sides as

$$4xh = 4x\left(\frac{1000}{x^2}\right) = \frac{4000}{x}$$

Hence the total surface area is given by

$$s(x) = 2x^2 + \frac{4000}{x} \qquad \text{for } x > 0.$$

Since the cost of production is proportional to the surface area, to minimize cost, we find the minimum value of s.

$$s'(x) = 4x - \frac{4000}{x^2} = \frac{4x^3 - 4000}{x^2} = 0$$

$$\Rightarrow \quad 4x^3 - 4000 = 0 \qquad \Rightarrow \qquad x^3 = 1000$$

$\Rightarrow$ x = 10 is the only critical point.

By the Second Derivative Test, we have

$$s''(x) = 4 + \frac{8000}{x^3} \quad \text{with} \quad s''(10) = 4 + 8 = 12 > 0$$

Thus x = 10 gives the minimum value s(10) = 600 sq. in.

The height is $h = \frac{1000}{x^2} = \frac{1000}{100} = 10$ in.

Hence the manufacturer would minimize the cost of production by manufacturing cubes of side 10 inches.

Curve Sketching

As a second application of the derivative we shall see here sketching the graphs of functions. You have been sketching the graphs of polynomial and rational functions starting from your high school mathematics. Here we systematically apply the notions of differential calculus to give precise meaning to the asymptotes, intervals of increase and decrease, the turning points and find the range of the functions.

First we shall list the important items that will help us in sketching the graph of a function y = f(x).

1) Determine the domain of the function f.
2) Find the intercepts of the function f.
   - x-intercepts are points of the form (x, 0)
   - y-intercepts are points of the form (0, y)
3) Determine the asymptotes, if any, of the function f.
   - A line x = c is a vertical asymptote of the graph of f iff

$\lim_{x\to c^+} f(x) = \pm\infty$ or $\lim_{x\to c^-} f(x) = \pm\infty$.

- A line y = L is a horizontal asymptote of the graph of f iff
  $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$.
- A line y = ax + b is an oblique (or skew) asymptote of the graph of f iff
  $\lim_{x\to\pm\infty}[f(x) - (ax + b)] = L$

4) Determine the intervals of monotonicity of the function f.
   - f is increasing for all x at which $f'(x) > 0$
   - f is decreasing for all x at which $f'(x) < 0$
5) Find extreme values of f, if any.
   Find the critical points of f and apply the first or second derivative tests to determine whether they are relative extreme points or not.
6) If necessary plot some additional points to help you see the behavior of the function.

Example 4.71: Sketch the graph of $f(x) = \frac{x-2}{x+2}$.

Solution. 2.1 The domain of f is **R**\{-2} and the x-intercept is the value of x for which

$f(x) = \frac{x-2}{x+2} = 0 \Rightarrow x = 2$. Hence x-intercept at (2, 0)

The y-intercept is the value of y when x = 0, i.e. $f(0) = \frac{0-2}{0+2} = -1$. Hence y-intercept at (0, -1).

Since $\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{x-2}{x+2} = -\infty$, the line x = -2 is a vertical asymptote to the graph of f.

Also you can check that $\lim_{x\to -2^-} f(x) = \infty$

Since $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x-2}{x+2} = \lim_{x\to\infty} \frac{x-2/x}{x+2/x} = 1$, the line y = 1 is a horizontal asymptote for the graph of f.

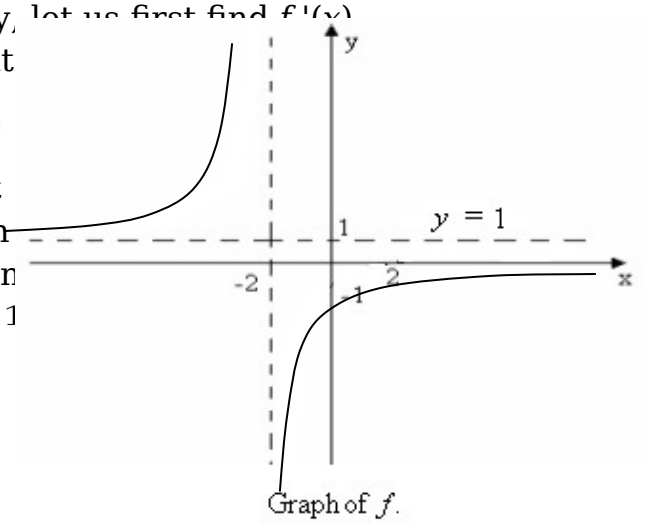
To find the intervals of monotonicity, let us first find $f'(x)$.

By the Quotient Rule for Differentiat

$f'(x) = (\frac{x-2}{x+2})' = \frac{(x+2)(x-2)' - (x-2)(x+2)'}{(x+2)^2} =$

Hence $f'(x) > 0$ for every element is strictly increasing on $(-\infty, -2)$ and on f has no critical number and hence n

Additional points: f(-1) = -3, f(1) = -1



Graph of f.

The graph of f is given in Figure 4.16. Figure 4.16

Example 4.72: Sketch the graph of f(x) = x + $\frac{1}{x}$, for x ≠ 0

Solution: Since f(x) = x + $\frac{1}{x}$ = $\frac{x^2+1}{x}$ ≠ 0 and since x ≠ 0 f has no intercepts.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x+\frac{1}{x}\right) = \infty$ and $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x+\frac{1}{x}\right) = -\infty$

The line x = 0 (the y-axis) is a vertical asymptote of f.

$\lim_{x\to\infty}[f(x) - x] = \lim_{x\to\infty}\frac{1}{x} = 0$

Hence the line y = x is an oblique asymptote of the graph of f.

$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2} = \frac{(x-1)(x+1)}{x^2} = 0$ gives two critical points x = 1 and x = -1.

Using a sign chart to find the intervals of monotonicity:

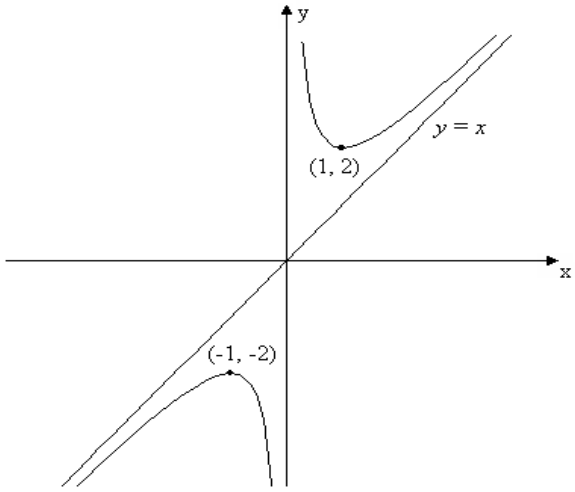
		-1		1	
x – 1	- - - - - - - - - - - - - - - - -	0	+ + + + + ++	+	+
x + 1	- - - - - - -	0	+ ++ + + + + + + + + + + + +		
$f'(x)$	+ + + + +	0	- - - - - - -	0	+ + + + + + + + +

$f'(x) > 0$ in the interval $(-\infty, -1) \cup (1, \infty)$ so that it is strictly increasing in $(-\infty, -1) \cup (1, \infty)$.

$f'(x) < 0$ in the interval $(-1, 1) \setminus \{0\}$ so that f is strictly decreasing in $(-1, 1) \setminus \{0\}$.

Using the first derivative test, you can see that f(-1) = -2 is a local maximum and f(1) = 2 is a local minimum. You can also apply the second derivative test to see this.

Additional points: f(-2) = $-5/2$, f$(-1/2) = \frac{-5}{2}$, f$(1/2) = 5/2$, f(2) = $\frac{5}{2}$.



Graph of $f(x) = x + \frac{1}{x}$

The graph is given in Figure 4.17. Figure 4.17

Related Rates

One of the most important applications of the derivative is to solve problems involving rates of change. As was mentioned at the beginning of this section the derivative measures the rate of change of a variable quantity (which is the independent variable x) with respect to another variable (which is the dependent variable y = f(x)). Here we shall apply this to solve some practical related rates problems.

Example 4.73: Suppose a particle P starts from a point 0 and moves along a straight line in the positive direction as See in Figure 4.18

Let s(t) devote the distance traveled from 0 in t seconds. If we assume that the speed P is constant, then we can compute the speed as

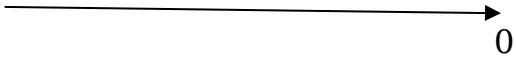


Figure 4.18

$$\text{speed} = \frac{\text{distance travele}}{\text{time elapsed}}$$

If we are interested to find the average speed of the particle between two times t_1 and t_2 (with $t_1 < t_2$), we get

$$\text{Average speed} = \frac{\text{change in distance}}{\text{change in time}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

In particular if t_1 is any time t and t_2 is a short time later say $t_2 = t + h$ for $h > 0$, then we have

Approximate speed (at t = t_1) = $\frac{s(t+h) - s(t)}{t+h-t} = \frac{s(t+h) - s(t)}{h}$

If the speed is not even constant, by taking h smaller and smaller we can approximate the speed of the particle at time t, to get what is called the (**instantaneous**) **velocity** of the particle as

$$v(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

Thus, if s(t) denotes the position function of the particle its velocity is given by

$v(t) = s'(t) = \frac{ds}{dt}$ - rate of change of position.

Similarly, the **acceleration** of the particle can be obtained by

$a(t) = v'(t) = \frac{dv}{dt}$ - rate of change of velocity

$$= \lim_{h \to 0} \frac{v(t+h) - v(t)}{h} = s''(t) = \frac{d^2s}{dt^2}$$

For instance if $s(t) = t^3 - 6t^2 + 20$ for $0 \le t \le 6$,
then $v(t) = s'(t) = 3t^2 - 12t$
and $a(t) = v'(t) = s''(t) = 6t - 12$

In general, if any quantity q is a function of time t, then the rate of change of the quantity with respect to time is given by the derivative q′(t).

Example 4.74 : Water is flowing into a vertical cylindrical tank of radius 2 feet at the rate of 8 ft^3/min. How fast is the water level rising after t minutes?

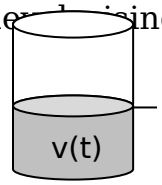


Figure 4.19

Solution: Let v(t) denote the volume of water in the tank after t minutes and let h(t) denote the height of water in the tank after t minutes. See Figure 4.19.

Since the rate at which water is flowing into the tank is 8 ft^3/min. the volume of water in the tank after t minutes is

$$v(t) = 8t$$

On the other hand since the base of the cylinder is 2 feet and height in h minutes is h(t), we have the volume

$$v(t) = \pi r^2 h(t) = 4\pi h(t)$$

Thus $4\pi h(t) = 8t \Rightarrow h(t) = \frac{2}{\pi}t$

The rate at which the water level is rising in then

$h'(t) = \frac{2}{\pi}$ ft|min, a constant! (why?)

Example 4.75: Two automobiles start from a point A at the same time. One travels west at 60 km/hr and the other travels north at 35 km/hr. How fast is the distance between them increasing 3 hrs later?

Solution: Let s(t) denote the distance between the two cars after t hrs. In t hrs the car due north travels 35t kms and the car due west travels 60t kms as seen in Figure 4.20

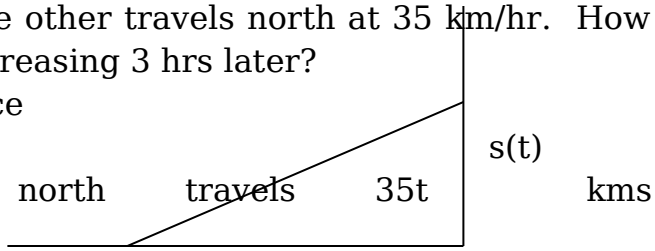


Figure 4.20.

Hence the distance s(t) between the two cars in t hrs is

$s(t) = \sqrt{(35t)^2 + (60t)^2}$

The rate of change of the distance between the cars is

$s'(t) = \frac{2(35)^2 t + 2(60)^2 t}{2\sqrt{(35t)^2 + (60t)^2}}$ … How ?

Hence after 3 hrs the distance between the two cars is increasing at the rate of

$s'(3) = \frac{3(35)^2 + 3(60)^2}{3\sqrt{(35)^2 + (60)^2}} = 5\sqrt{193}$ km/hr

Exercise 4.3

1. Find relative extrema and the intervals in which the given function is increasing or decreasing

   a) $f(x) = 5 - 4x - x^2$ b) $g(x) = x^3 + x^2 - x - 4$

   c) $f(x) = \frac{x}{x^2+1}$ d) $g(x) = x^2 + \frac{1}{x^2}$

2. Use the First or Second Derivative Test to determine relative extreme values of the function

   a) $f(x) = 5x^2 - 2x + 1$ b) $g(x) = \frac{x^2}{4} + \frac{4}{x}$

   c) $f(x) = x^4 + \frac{1}{2}x$ d) $g(x) = \frac{1}{x^2+1}$

e) $f(x) = \frac{\cos x}{1+\sin x}$ f) $g(x) = (x^2 + 2)^6$

3. Sketch the graph of the following functions

a) $f(x) = (x^2 - 1)^2$ b) $g(x) = \frac{e^x}{x}$ c) $g(x) = \frac{x^3 - 3x^2 + 4}{x^2 - 1}$

4. A menu of total area of 100 sq. in. is printed with 2 in. margins at the top and bottom and 1in. margins at the sides. For what dimensions of the menu is the printed area largest?
5. A rectangle of perimeter p is rotated about one of its sides so as to from a cylinder. Of all such possible rectangles, which generated a cylinder of maximum volume?
6. The volume of a spherical balloon is increasing at a constant rate of 8 cubic feet per minute. How fast is the radius of the sphere increasing when the radius is exactly 10 feet?
7. At midnight ship B was 90 miles due south of ship A. Ship A sailed east at 15 m/hr and ship B sailed north at 20 m/hr. At what time were they closest to each other?

4.4. Integrals and their applications

In this section we shall introduce the second major part of calculus known as integral calculus. Just like subtraction is the inverse process of addition, integration is the inverse process of taking the derivative of a function. Historically, integral calculus was developed in solving problems connected with finding areas of regions with curved boundaries.

Section Objectives

At the end of this section you should be able to:

- define an anti-derivative of a continuous function.
- state properties of anti-derivatives.
- find indefinite integrals of some elementary functions.
- evaluate the integrals of functions using the techniques of integration.
- solve integrals involving trigonometric functions.
- find the definite integral of continuous functions.
- apply the concepts of definite integrals to find areas of regions bounded by continuous functions.

The Indefinite Integral

As is mentioned above the process of integration is the inverse process of differentiation and hence is sometimes called taking anti-derivatives.

Definition 4.13: A function F(x) is called an **anti-derivative** of a continuous function f(x) if and only if $F'(x) = f(x)$ for every x in the domain of f.

Example 4.76: Let $f(x) = 3x^2 + 4x$. Then the function $F_1(x) = x^3 + 2x^2$ is an anti-derivative of f(x), since $F'(x) = \frac{d}{dx}(x^3 + 2x^2) = 3x^2 + 4x = f(x)$.

Note that F_1 is not the only anti-derivative of f(x). You can also check that $F_2(x) = x^3 + 2x^2 + 5$ and $F_3(x) = x^3 + 2x^2 - 7$ are also anti-derivatives of f. In fact, if c is any real number, then $F(x) = x^3 + 2x^2 + c$ is an anti-derivative of $f(x) = 3x^2 + 4x$ since $F'(x) = \frac{d}{dx}(x^3 + 2x^2 + c) = 3x^2 + 4x = f(x)$

Theorem 4.19: If F(x) is an anti-derivative of f(x), then F(x) + c, where c is an arbitrary constant, is also an anti-derivative of f(x).

Notation and terminologies: Given a function f, the symbol $\int f(x)dx$ stands for any (and hence all) anti-derivatives of f. i.e. if F(x) is an anti-derivative of f(x), we write $\int f(x)dx = F(x) + c$, for any constant c. The symbol $\int$ is called the **integral sign.** The function f(x) is called the **integrand**, x is called the **variable of integration**, and c is called a **constant of integration.**

$\int f(x)dx$ is also called the **indefinite integral** of f with respect to x.

Examples 4.77: We have

a) $\int 3x^2dx = x^3 + c$

b) $\int e^xdx = e^x + c$

c) $\int \sec^2xdx = \tan x + c$

d) $\int \sin xdx = -\cos x + c$;

e) $\int \frac{1}{x}dx = \ell n|x| + c$

f) $\int \csc x \cot xdx = -\csc x + c$

g) $\int \cos x\,dx = \sin x + c$

- **Properties of the Indefinite Integral**

Suppose F and G are antiderivatives of f and g, respectively, and k is a constant. Then

1) $\int kf(x)dx = k\int f(x)dx = kF(x) + c.$

2) $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx = F(x) + G(x) + c.$

3) $\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx = F(x) - G(x) + c.$

Examples 4.78:

1) $\int 4\cos x dx = 4\int \cos x dx = 4\sin x + c$

2) $\int\left(e^x - \frac{1}{x}\right)dx = \int e^x dx - \int\frac{1}{x}dx = e^x - \ell n|x| + c$

3) If $f(x) = x^r$, for any rational $r \neq -1$, then

$\int f(x)dx = \int x^r dx = \frac{x^{r+1}}{r+1} + c$ (verify!)

Thus, $\int x^5 dx = \frac{x^6}{6} + c$ and $\int x^{-3/2}dx = \frac{x^{-1/2}}{-1/2} + c = \frac{-2}{\sqrt{x}} + c.$

4) $\int\left(\frac{1}{x^3} + 2\sec^2 x\right)dx = \int x^{-3}dx + 2\int \sec^2 x dx$

$= -\frac{1}{2x^2} + 2\tan x + c.$

5) If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ is a polynomial, then its anti-derivative is given by

$$P(x) = \int P(x)dx = \frac{a_n x^{n+1}}{n+1} + \frac{a_{n-1}x^n}{n} + \ldots + \frac{a_2x^3}{3} + \frac{a_1x^2}{2} + a_0x$$

Thus, $\int (3x^4 + \sqrt{2}x^3 - 5x + 2)dx = 3\int x^4dx + \sqrt{2}\int x^3dx - 5\int xdx + \int 2dx$

$= \frac{3}{5}x^5 + \frac{\sqrt{2}}{4}x^4 - \frac{5}{2}x^2 + 2x + c$

- **Some Techniques of Integration**

In the previous section we were trying to find anti-derivatives of some functions whose derivatives can easily be found from the previous unit on differentiation. But there are various functions such as

$f(x) = (x + 3)^5$, $g(x) = xe^{-x}$ and $h(x) = \frac{2x}{x(x^2 - 4)}$

whose anti-derivatives are not readily found. In this section we shall see some techniques to find the integrals of such functions.

a) Integration by Substitution

This technique is basically developed by reversing the Chain Rule. It is very helpful in finding the integrals of functions that appear as the composite of two functions.

Suppose we want to find the indefinite integral

$$\int (x+3)^5 dx$$

we may expand $(x + 3)^5$ and then integrate term by term using the formula

$$\int x^r dx = \frac{1}{r+1} x^{r+1} + c.$$

But this would obviously be very tedious and cumbersome. On the other hand if we replace or substitute u for x + 3, we get

$(x + 3)^5 = u^5$ and $\frac{du}{dx} = \frac{d}{dx}(x + 3) = 1 \Rightarrow dx = du.$

Thus, $\int (x+3)^5 dx = \int u^5 du = \frac{1}{6}u^6 + c$. Hence, $\int (x+3)^5 dx = \frac{1}{6}(x + 3)^6 + c$, for some constant c.

Theorem 4.20: If g′(x) is continuous $\forall x \in [a, b]$ and f is continuous at g(x), then

$\int f(g(x))g'(x)dx = \int f(u)du$ - **Integration by Substitution**

Example 4.79: Evaluate $\int 2x(x^2 - 5)^6 dx$

Solution: Let $u = x^2 - 5$. Then, $\frac{du}{dx} = 2x$ which implies that $du = 2xdx$. Thus,

$$\int 2x(x^2 - 5)^6 dx = \int u^6 du = \frac{1}{7}u^7 + c \text{ and hence } \int 2x(x^2 - 5)^6 dx = \frac{1}{7}(x^2 - 5)^7 + c.$$

Example 4.80: Integrate $\int \frac{x}{\sqrt{1+x^2}} dx$

Solution: Let $u = 1 + x^2$. Then, $\frac{du}{dx} = 2x$ which implies that $xdx = \frac{1}{2}du$. Therefore,

$$\int \frac{x}{\sqrt{1+x^2}} dx = \int \frac{1}{\sqrt{u}} \frac{du}{2} = \frac{1}{2}\int u^{-1/2} du = \frac{1}{2}.2.u^{1/2} + c = \sqrt{u} + c = \sqrt{1+x^2} + c.$$

Example 4.81: Integrate $\int \sin 4x\,dx$

Solution: Let u = 4x. Then $\frac{du}{dx} = 4 \Rightarrow dx = \frac{1}{4}du$. Thus $\int \sin 4x\,dx = \frac{1}{4}\int \sin u\,du = -\frac{1}{4}\cos u + c = -\frac{1}{4}\cos 4x + c$.

In general $\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$, and $\int \cos bx\,dx = \frac{1}{b}\sin bx + c$

These two formulas can be used to find integrals involving trigonometric functions together with trigonometric identities.

Example 4.82: Integrate $\int \sin^2 x \cos^2 x\,dx$

Solution: From trigonometric identities we have

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Thus $\int \sin^2 x \cos^2 x\,dx = \int \frac{1}{2}(1 - \cos 2x)\frac{1}{2}(1 + \cos 2x)dx$

$$= \frac{1}{4}\int (1 - \cos^2 2x)dx = \frac{1}{4}\int \left[1 - \frac{1}{2}(1 + \cos 4x)\right]dx$$

$$= \frac{1}{8}\int dx - \frac{1}{8}\int \cos 4x\,dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + c.$$

Example 4.83: Find $\int \tan x\,dx$

Solution: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$. Let u = cos x. Then $\frac{du}{dx} = -\sin x \Rightarrow \sin x\,dx = -du$.

Hence, $\int \tan x\,dx = \int \frac{-du}{u} = -\int \frac{1}{u}du = -\ell n|u| + c = -\ell n|\cos x| + c$.

You can similarly find $\int \cot x\,dx$.

Example 4.84: Integrate $\int e^{-2x}dx$

Solution: Let u = - 2x. Then $\frac{du}{dx} = -2 \Rightarrow dx = -\frac{1}{2}du$.

So that $\int e^{-2x}dx = -\frac{1}{2}\int e^u du = \frac{1}{2}e^u + c = \frac{1}{2}e^{-2x} + c$

In general since $\frac{d}{dx}e^{f(x)} = f'(x)\,e^{f(x)}$, *we have* $\int f'(x)e^{f(x)}dx = e^{f(x)} + c$.

Thus $\int xe^{x^2}dx = \frac{1}{2}e^{x^2} + c$ and $\int 3x^2e^{x^3}dx = e^{x^2} + c$.

b) Integration by Parts

The method of integration by parts is basically developed from the Product Rule for differentiation. If f and g are differentiable functions, we have

$(f(x)g(x))' = f'(x)\,g(x) + g'(x)\,f(x)$

Integrating on both sides with respect to x, we get

$f(x)\,g(x) = \int f'(x)g(x)dx + \int g'(x)f(x)dx$

If one of the integrals on the right can be easily evaluated, we can find the other integral using the following theorem

Theorem 4.21: If f and g are differentiable functions, then

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Ex

So

Integration by parts

e^x.

Therefore,

$\int xe^x dx = xe^x - \int e^x.1.dx = xe^x - e^x + c$, for some constant c.

Integration by parts can be easily remembered using the following substitutions.

Let $u = f(x)$ and $v = g(x)$

Then $du = f'(x)dx$ and $dv = g'(x)dx$

So that $\int f(x)g'(x)dx = \int udv = f(x)g(x) - \int g(x)f'(x)dx = uv - \int vdu$

Thus, $\int udv = uv - \int vdu$ **- Integration by Parts**.

Example 4.86: Find $\int x\ell nxdx$

Solution: Let $u = \ell nx$, $dv = xdx$. Then, $du = \frac{1}{x}dx$, $v = \frac{x^2}{2}$.

Thus $\int x\ell nxdx = \frac{x^2}{2}\ell nx - \int\frac{x^2}{2}\frac{1}{x}dx = \frac{x^2}{2}\ell nx - \int\frac{x}{2}dx = \frac{x^2}{2}\ell nx - \frac{x^2}{4} + c$.

Example 4.87: Find $\int \ell nxdx$

Solution: Let $u = \ell nx$ and $dv = dx$. Then, $du = \frac{1}{x}dx$ and $v = x$

Hence $\int \ell nxdx = x\ell nx - \int x.\frac{1}{x}dx = x\,\ell nx - \int dx = x\ell nx - x + c = x(\ell nx - 1) + c$.

Example 4.88: Find the integral $\int x^2e^x dx$

Solution: Let $u = x^2$ and $dv = e^x dx$. Then, $du = 2xdx$, $v = e^x$ and $\int x^2 e^x dx$ $= x^2e^x - \int 2xe^x dx$.

But we have seen above that $\int xe^x dx = xe^2 - e^x + c$. Hence,

$\int x^2 e^x dx = x^2e^x - 2(xe^x - e^x + c) = e^x(x^2 - 2x + 1) + c_1$ where $c_1 = -2c$ is a constant.

In some cases we may have to apply integration by parts more than once to arrive at the required result as in the following example.

Example 4.89: Find $\int e^x \cos x dx$

Solution: Let $u = e^x$ and $dv = \cos x dx$. Then, $du = e^x dx$ and $v = \sin x$. Thus,

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx.$$

To evaluate the integral on the right, we again use integration by parts.

Let $u = e^x$ and $dv = \sin x\, dx$. Then, $du = e^x dx$ and $v = -\cos x$.

Thus, $\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$ which implies $2\int e^x \cos x dx = e^x \sin x + e^x \cos x + c$.

Therefore, $\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) + c$.

c) Integration by the Method of Partial Fractions

The method of Partial Fractions is used for rational functions

$$f(x) = \frac{p(x)}{q(x)}$$

where degree of p(x) is less than degree of q(x). (If not we can apply long division to write f(x) as a sum of a polynomial and a rational function with the desired property.) The first step in this method is to factorize the denominator q(x) into linear factors, if possible. (The case where we have irreducible quadratic factors of q(x) will not be treated here.) Now with each linear factor $(ax + b)^m$ (of multiplicity m) we associate constants A_1, A_2, ...,A_m and write $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} +$...with the A_i's to be determined. Then, the rational function f(x) is then expressed as a sum of simple rational functions and can be easily integrated.

Example 4.90: Find $\int \frac{1}{x^2 - 4} dx$

Solution: By factorizing $x^2 - 4$ as $(x - 2)(x + 2)$, we have

$$\frac{1}{x^2 - 4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2) + B(x-2)}{(x-2)(x+2)}$$

Since the denominators are equal, we equate the numerators as A(x+2) + B(x - 2) = 1.

From equality of polynomials, we get

$\left.\begin{matrix} A + B = 0 \\ 2A - 2B = 1 \end{matrix}\right\} \Rightarrow$ A = 1/4 and B = $\frac{-1}{4}$

Hence, $\int \frac{1}{x^2 - 4} dx = \int \left(\frac{1/4}{x-2} + \frac{-1/4}{x+2} \right) dx = \frac{1}{4} \int \frac{dx}{x-2} - \frac{1}{4} \int \frac{dx}{x+2} = \frac{1}{4} \ell n|x - 2| - \frac{1}{4} \ell n|x+2| + c.$

Example 4.91: Find $\int \frac{3x^2 + x - 1}{x^3 - x^2 - 2x} dx$

Solution: The denominator $x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x + 1)(x - 2)$ has three roots 0, -1 and 2.

$$\frac{3x^2 + x - 1}{x^3 - x^2 - 2x} = \frac{3x^2 + x - 1}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$= \frac{A(x-1)(x-2) + Bx(x-2) + Cx(x-1)}{x(x+1)(x-2)}$$

$\Rightarrow$ A(x+1) (x - 2) + B x(x - 2) + Cx(x+1) = $3x^2 + x - 1$

This equation is true for all $x \in \mathbf{R}$. In particular,

when x = 0, A(1)(-2) = -1 $\Rightarrow$ A = 1/2

when x = -1, B(-1)(-3) = 1 $\Rightarrow$ B = 1/3

when x = 2, C(2)(3) = 13 $\Rightarrow$ C = 13/6

Hence, $\int \frac{3x^2 + x - 1}{x^3 - x^2 - 2x} dx = \frac{1}{2} \int \frac{dx}{x} + \frac{1}{3} \int \frac{dx}{x+1} + \frac{13}{6} \int \frac{dx}{x-2} = \frac{1}{2} \ell n|x| + \frac{1}{3} \ell n|x + 1| + \frac{13}{6} \ell n|x-2| + c.$

• The Definite Integral

For a very long time, mathematicians have struggled with the problem of finding areas of plane regions. Until the invention of the integral calculus, however, the regions considered were mostly those regions bounded by straight lines, called polygons, with a few exceptions such as the circle and the ellipse. The Greek mathematicians found the area of a polygon by first finding the area of a rectangle, then finding the area a parallelogram, and then finding the area of a triangle. The area of a polygon can be used to approximate the area of a region bounded by curved boundaries. For

instance, the area of a circle can be found by drawing a sequence of inscribed polygons
P_4, P_8, P_{16}, …,P_n, and then taking limit as $n \to \infty$.

To develop the idea for more general regions, consider the region bounded by the graphs of

$$y = 2x^2 + 1, \quad x = 0, \quad x = 6 \quad \text{and} \quad \text{x-axis.}$$

To find the area of the region, let us identify the region S by drawing its boundaries, namely the graphs of $y = f(x) = 2x^2 + 1$, $x = 0$, $x = 6$ and the x-axis as shown in Figure 4.21.

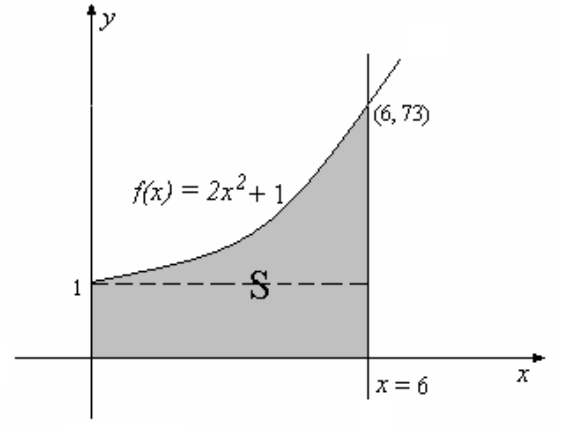


Figure 4.21

Unfortunately, since $f(x) = 2x^2 + 1$ is a curve that is not a line segment, we cannot find the area of the region by the elementary methods. So, it is necessary to develop a stronger technique that also generalizes the elementary method and enables us to find the area of such regions.
Let A(S) denote the area of the region S. It is not difficult to give lower and upper bounds of A(S). For instance, we consider the rectangle r that is enclosed by the boundaries of S and the rectangle R that encloses S, as shown in Figure 4.22.

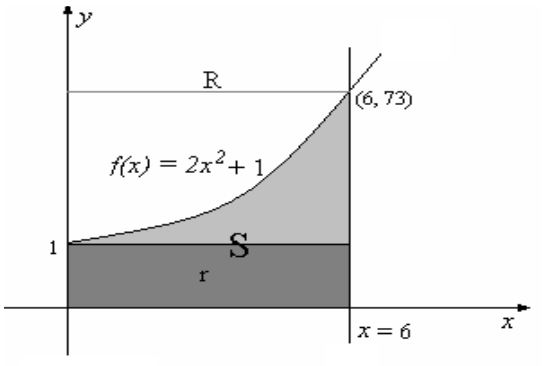


Figure 4.22

Then A(r) = 6 and A(R) = 6×73 = 438. Hence $6 \leq A(S) \leq 438$, which gives a wide range of bounds of A(S).

Better bounds of A(S) can be obtained if we consider the finer rectangles r_1, r_2, r_3, r_4, r_5 and r_6 that are enclosed by the boundaries of S and R_1, R_2, R_3, R_4, R_5 and R_6 that enclose S as shown in Figure 4.23.

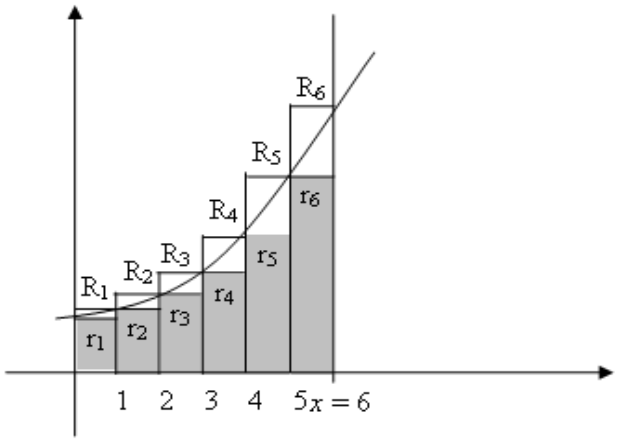


Figure 4.23

Evidently, each of the rectangles has base 1 unit but varying heights. It follows that

$$A(r_1)+A(r_2) + \ldots + A(r_6) \leq A(S) \leq A(R_1) + A(R_2) + \ldots + A(R_6)$$

i.e., $\sum_{i=1}^{6} A(r_i) \leq A(S) \leq \sum_{i=1}^{6} A(R_i)$ which gives $116 \leq A(S) \leq 188$.

To give a formal definition of the subdivisions, for any positive integer n, divide [a, b] into subintervals by introducing points of subdivision $x_0, x_1, \ldots x_n$

Definition 4.14: A **partition** of [a, b] is a finite set P of points $x_0, x_1, \ldots, x_n$ such that

$a = x_0 < x_1 < x_2 < \ldots < x_n = b$. We describe P by writing $P = \{ x_0, x_1, \ldots x_n\}$

By definition, any partition of [a, b] must contain a and b.

The length of any subinterval $[x_{i-1}, x_i]$ of a partition P is defined and given by

$$\Delta x_i = x_i - x_{i-1}$$

In particular, when the lengths of each subintervals are equal, it is called a **regular** partition.

In this section we shall consider only regular partitions, so that the length of each subinterval is

$$\Delta x_i = x_i - x_{i-1} = \frac{b-a}{n}$$

Having chosen a partition P of [a, b], we inscribe and circumscribe rectangles on the region R using the division points of P as seen in Figure 7.3(a) and (b). Since f is continuous on [a, b], by the Maximum-Minimum Theorem, for each i between 1 and n, there is a minimum value m_i and a maximum value M_i of f on the subinterval $[x_{i-1}, x_i]$. If r_i and R_i denote the inscribed and circumscribed rectangles on $[x_{i-1}, x_i]$, respectively, then the area of r_i is $A(r_i) = m_i\Delta x_i$ and the area of R_i is $A(R_i) = M_i\Delta x_i$, since the base of both r_i and R_i is $\Delta x_i = x_i - x_{i-1}$. From our observation in Figure 7.3 (a) and (b) we see that the area of the region R is between the sum of the inscribed rectangles and the sum of the circumscribed rectangles.

Definition 4.15: Let f be continuous on [a, b] and P be any partition of [a, b].

The sum $L_f(P) = m_1\Delta x_1 + m_2\Delta x_2 + \ldots + m_n\Delta x_n$

is called the **lower sum** of f associated with P and the sum

$$U_f(P) = M_1\Delta x_1 + M_2\Delta x_2 + \ldots + M_n\Delta x_n$$

is called the **upper sum** of f associated with P.

From our construction we see that if P is any partition of [a, b], then the area of R should be between $L_f(P)$ and $U_f(P)$ i.e.

$$L_f(P) \le \text{Area } (R) \le$$

Example 4.92: Let $f(x) = x^2$ for $o \le x \le 2$ and let $P = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$ be a partition of [0, 2].

Then the subdivision of [0, 2] associated with P are $\left[0, \frac{1}{2}\right]$, $\left[\frac{1}{2}, 1\right]$, $\left[1, \frac{3}{2}\right]$, $\left[\frac{3}{2}, 2\right]$. Since x^2 is an increasing function on [0, 2], the minimum value of f on each subinterval is at the left end point and the maximum value of f at the right end point. Thus

$$m_1 = f(0) = 0, \qquad m_2 = f\left(\frac{1}{2}\right) = \frac{1}{4}, \qquad m_3 = f(1) = 1, \qquad m_4 = f\left(\frac{3}{2}\right) = \frac{9}{4}$$

and $M_1 = f\left(\frac{1}{2}\right) = \frac{1}{4}$, $M_2 = f(1) = 1$, $M_3 = f\left(\frac{3}{2}\right) = \frac{9}{4}$, $M_4 = f(2) = 4$

The base of each subinterval is $\Delta x_i = \frac{2-0}{4} = \frac{1}{2}$. Thus the lower sum of f associated with P is

$$L_f(P) = 0.\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} + 1.\frac{1}{2} + \frac{9}{4}.\frac{1}{2} = \frac{7}{4}$$

and the upper sum of f associated with P is

$$U_f(P) = \frac{1}{4}.\frac{1}{2} + 1.\frac{1}{2} + \frac{9}{4}.\frac{1}{2} + 4.\frac{1}{2} = \frac{15}{4}$$

Therefore the area of the region R below the graph of $f(x) = x^2$ above the x-axis on [0, 2] is between $\frac{7}{4}$ and $\frac{15}{4}$, i.e. $\frac{7}{4} \le \text{Area}(R) \le \frac{15}{4}$.

Definition 4.16: Let f be continuous on [a, b]. The **definite integral of f from a to b** is the unique number I satisfying $L_f(P) \le I \le U_f(P)$ for every partition P of [a, b].

This integral is denoted by

$$I = \int_a^b f(x)dx$$

The numbers a and b are called the **lower** and **upper limits of integration**, respectively.

Note that as the number of subdivisions of an interval [a, b] increases, the minimum and the maximum values of f on $[x_{i-1}, x_i]$ are close to each other. For each i from 1 to n if we take an arbitrary number t_i in $[x_{i-1}, x_i]$, then we get the sum

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \ldots + f(t_n)\Delta x_n$$

This sum is called a **Riemann sum** or an **Integral sum.**

Even though it is sometimes possible to calculate $\int_a^b f(x)dx$ by finding formulas for lower and upper sum we are to evaluate it here by the use of the Fundamental Theorem of Calculus.

For the moment we can conclude that if f is continuous and nonnegative on [a, b], then the area of the region R between the graph of f and the x-axis on [a, b] is given by

$$\text{Area}(R) = \int_a^b f(x)dx.$$

Remark: *The definite integral has the following properties.*

If f and g are integrable over [a, b] and k is a constant, then

a) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

b) $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

c) *If f(x) ≥ 0, for a ≤ x ≤ b, then* $\int_a^b f(x)dx \geq 0$ *and*

if f(x) ≤ 0, for a ≤ x ≤ b, then $\int_a^b f(x)dx \leq 0$.

d) *If m ≤ f(x) ≤ M for all x ∈ [a, b], then*

$$m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$$

e) *If c is any number in (a, b), then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$ *- Additive Property*

f) $\int_a^a f(x)dx = 0$ *for any number a.*

g) $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

To develop a general method for evaluating $\int_a^b f(x)dx$ without computing lower and upper sums we shall state the most important theorem in calculus: **The Fundamental Theorem of Calculus**. To this end let f(t) be continuous on [a, b]. Then f is integrable on [a, b] and for any x∈[a, b] the definite integral $\int_a^x f(x)dt$ exists. Define a function F on [a, b] as F(x) = $\int_a^x f(t)dt$

In effect the Fundamental Theorem of Calculus states that the function F(x) is differentiable with derivative f(x) thereby eliminating the integral by the derivative. It also shows us how to evaluate the definite integral.

Theorem 4.22: (**Fundamental Theorem of Calculus**)
Let f(t) be continuous on [a, b] and for each x ∈[a, b] let

$$F(x) = \int_a^x f(t)dt$$

Then (i) F(x) is a differentiable function with F′(x) = f(x)

(ii) If F is any anti-derivative of f on [a, b], then $\int_a^b f(t)dt$ = F(b) –

Remarks: *a) From (ii) to evaluate* $\int_a^b f(x)dx$ *all we have to do is to find an anti-derivative of F of f and find the difference of its values at a and at b. This is usually denoted by* $[F(x)]_a^b$ *or* $F(x)\Big|_a^b$ *to mean F(b) – F(a).*

b) If F is an anti-derivative of f, then F(x) + c, for any constant c is also an anti-derivative of f. But since $[F(x)+c]_a^b$ *= (F(b) + c) – (F(a) + c) = F(b) – F(a) =* $F(x)\Big|_a^b$

the constant c does not play any role in evaluating the definite integral. Thus we can always take c = 0.

Example 4.93: Let f(x) = x^2 for 0 ≤ x ≤ 2. Then F(x) = $\frac{1}{3}x^3$ is an anti-derivative of f, so that by the Fundamental Theorem of Calculus.

$$\int_0^2 x^2dx = \frac{1}{3}x^3\Big|_0^2 = F(2) - F(0) = \frac{1}{3}2^3 - \frac{1}{3}.0^3 = \frac{8}{3} - 0 = \frac{8}{3}$$

From our previous discussion, the area of the region R under the graph of f(x) = x^2 on [0, 2] above the x-axis is thus $\int_0^2 x^2dx = 8/3$ sq. units.

Example 4.94: Evaluate each of the following definite integrals

a) $\int_1^4 3\sqrt{x}dx$ b) $\int_0^\pi \sin x dx$

c) $\int_0^{\pi/2}(x+\cos x)dx$ d) $\int_0^1 (5x^3 + 2x - e^x)dx$

Solution: a) Since F(x) = $3.\frac{2}{3}.x^{3/2} = 2x\sqrt{x}$ is an anti-derivative of f(x) = $3\sqrt{x}$, we have

$$\int_0^4 3\sqrt{x}dx = F(4) - F(1) = 2(4)\sqrt{4} - 2(1)\sqrt{1} = 16\text{-}2 = 14$$

b) An anti-derivative of sinx is –cosx. Thus

$$\int_0^\pi \sin x dx = -\cos x\Big|_0^\pi = -\cos\pi - (-\cos 0) = -(-1) + 1 = 2.$$

c) $\int_0^{\pi/2}(x+\cos x)dx = \left[\frac{x^2}{2} + \sin x\right]_0^{\pi/2} = \left(\frac{\pi^2}{8} + 1\right) - (0 + 0) = \frac{\pi^2}{8} + 1.$

d) $\int_0^1 (5x^3 + 2x - e^x)dx = \left[\frac{5}{4}x^4 + x^2 - e^x\right]_0^1 = \left(\frac{5}{4} + 1 - e\right) - (0 + 0 - 1) = \frac{13}{4} - e$

Remark: *For functions that are given by more than one formula we evaluate the definite integral using the additive property.*

Example 4.95: Evaluate $\int_{-2}^{1} |x+1|\,dx$

Solution: By definition |x + 1| = $\begin{cases} x+1, & for\ x \geq -1 \\ -(x+1), & for\ x < -1 \end{cases}$

Then by Additive Property, we have

$$\int_{-2}^{1} |x+1|\,dx = \int_{-2}^{-1} -(x+1)\,dx + \int_{-1}^{1} (x+1)\,dx$$

$$= -\left[\frac{x^2}{2}+x\right]_{-2}^{-1} + \left[\frac{x^2}{2}+x\right]_{-1}^{1}$$

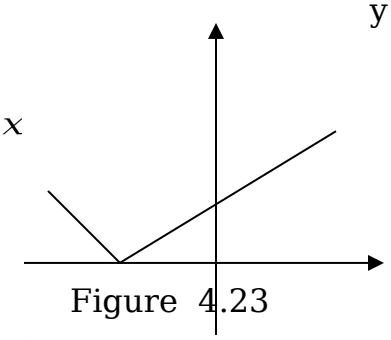


Figure 4.23

$$= -\left[\left(\frac{1}{2}-1\right)-(2-2)\right] + \left[\left(\frac{1}{2}+1\right)-\left(\frac{1}{2}-1\right)\right] = \frac{1}{2}+2 = 5/2$$

From the method of Integration by Substitution we have

$$\int f(g(x))g'(x)dx = \int f(u)du \text{ where u = g(x)}$$

If we are to evaluate this integral between a and b, we have,
when x = a, u = g(a) and when x = b, u = g(b). Thus it follows

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

- Change of Variable.

Example 4.96: Evaluate $\int_2^3 x\sqrt{x^2-4}\,dx$

Solution: We have two possibilities to evaluate such a definite integral. One way is to find an anti-derivative of $x\sqrt{x^2-4}$ and evaluate it between 2 and 3 by the Fundamental Theorem of Calculus. The other is to use the change of variable formula and change the limits of integration before integrating.
To this end, let u = g(x) = x^2 – 4. Then du = 2xdx.
When x = 2, u = g(2) = 0 and when x = 3, u = g(3) = 5

Thus $\int_2^3 x\sqrt{x^2-4}\,dx = \int_0^3 \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int_0^5 u^{1/2}du$

$$= \frac{1}{2}\cdot\frac{2}{3}u\sqrt{u}\Big|_0^5 = \frac{1}{3}\cdot 5\sqrt{5} - 0 = \frac{5\sqrt{5}}{3}.$$

- **Application of the Definite Integral: Area**

The definite integral has several applications such as finding areas of regions, arc length of curves, surface areas and volumes of solids of revolution. In this section we shall see how to find areas of plane regions with curved boundaries using the definite integrals.

In the previous section we have seen that if f(x) ≥ 0 for all x ∈[a, b] and if f is continuous on [a, b], then $\int_a^b f(x)dx$ gives the area of the region R below the graph of f, above the x-axis, between the lines x = a and x = b. For instance, if f(x) = x^2 for 0 ≤ x ≤ 2, then the area of R as given in is given by A(R) = $\int_0^2 x^2 dx = \frac{x^3}{3}\Big|_0^2 = 8/3$ sq. units.

- If f(x) ≤ 0 on [a, b], then taking g(x) = -f(x) ≥ 0 for a ≤ x ≤ b, the area of the region R below the x-axis, above the graph of f on [a, b] is given by A(R) = $\int_a^b g(x)dx = \int_a^b -f(x)dx = -\int_a^b f(x)dx$.

For instance, if f(x) = 2x for -2 ≤ x ≤ 0, then the area of the region R below the x-axis, above the graph of f on [-2, 0] is given by

$$A(R) = -\int_{-2}^{0} f(x)dx = -\int_{-2}^{0} 2x\,dx = -x^2\Big|_{-2}^{0} = -[0 - 4] = 4 \text{ sq. units.}$$

Now let f and g be continuous on [a, b], and assume that f(x) ≥ g(x) for a ≤ x ≤ b. Then the area of the region R below the graph of f, above the graph of g, and between the lines x = a and x = b is given by

$$A(R) = \int_a^b [f(x) - g(x)]dx$$

Example 4.97: Find the area of the region bounded by $f(x) = 2\sqrt{x}$, $g(x) = -x$ and line $x = 9$.

Solution. Sketching the graphs of $y = f(x)$, $y = g(x)$ and $x = 9$, the region R can be identified as shown in Figure 4.24.

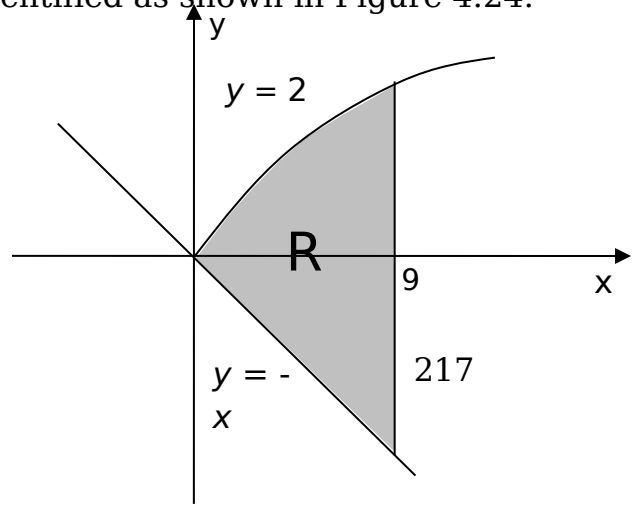


Figure 4.24

It follows that

$$A(R) = \int_0^9 (f(x) - g(x))dx = \int_0^9 (2\sqrt{x} + x)dx = \frac{4}{3}x^{\frac{3}{2}} + \frac{x^2}{2}\Big|_0^9 = 76.5.$$

Exercise 4.4

1. Evaluate the following indefinite integrals

   a) $\int (x^3 + 5)dx$

   b) $\int 2x^{-8}dx$

   c) $\int 3\sin x dx$

   d) $\int (4 - x + 3x^2 - 2x^5)dx$

   e) $\int (\cos x - 4e^x)dx$

   f) $\int \frac{x^3 - 4\sqrt{x}}{x^2}dx$

2. Find the following integrals by substitution

   a) $\int \frac{dx}{x - 3}$

   b) $\int e^{x/3}dx$

   c) $\int \sin^2 x \cos x dx$

   d) $\int \frac{\ell n^4 x}{x}dx$

3. Find the following integrals by the method of Integration by Parts.

   a) $\int x\cos x dx$

   b) $\int (x+1)3^x dx$

   c) $\int \frac{\ell n x}{x^2}dx$

   d) $\int x^2 \sin x dx$

4. Integrate the following by the method of Partial Fractions

   a) $\int \frac{dx}{(x+2)(3x+4)}$

   b) $\int \frac{x}{x^2 - x - 6}$

   c) $\int \frac{dx}{(x-1)(x-2)(x-3)}$

   d) $\int \frac{2x}{(x-2)^2}dx$

5. Find the area of the region R between the graph of f and the x-axis on the given interval

   a) f(x) = $x^2 + 1$, on [1, 3]

   b) f(x) = 2 + cosx, on $[0, 3\pi/2]$

   c) f(x) = $\frac{1}{x}$, on [1, 4]

d) $f(x) = |x| - 1$, on $[-1, 2]$

6. Find the area of the region between the graphs of the following functions.

. a) $f(x) = x^2$ and $g(x) = 2 – x$

b) $f(x) = e^x$, $x = -1$, $x = 3$ and the x-axis

c) $f(x) = x^2 – 4$ and $g(x) = 4 – x^2$