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Combinatorics and Complexity of Partition Functions

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Contents

Chapter 1
Introduction

What this book is about. What is a partition function?

The answer depends on who you ask. You get one (multi)set of answers if you ask physicists, and another (multi)set if you ask mathematicians (we allow multisets, in case we want to account for the popularity of each answer). In this book, we adopt a combinatorial view of partition functions. Given a family $\mathcal{F}$ of subsets of the set $\{1, \ldots, n\}$, we define the *partition function* of $\mathcal{F}$ as a polynomial in n real or complex variables $x_1, \ldots, x_n$,

$$p_{\mathcal{F}}(x_1, \ldots, x_n) = \sum_{S \in \mathcal{F}} \prod_{i \in S} x_i. \tag{1.1}$$

Under typical circumstances, it is unrealistic to try to write $p_{\mathcal{F}}$ as a sum of monomials explicitly, for at least one of the following two reasons:

(1) the family $\mathcal{F}$ is very large
or
(2) we are not really sure how large $\mathcal{F}$ is and it will take us a while to go over all subsets S of $\{1, \ldots, n\}$ and check whether $S \in \mathcal{F}$.

Typically, however, we will have no trouble checking if any particular subset S belongs to $\mathcal{F}$. A good example is provided by the family $\mathcal{H}$ of all Hamiltonian cycles in a given graph G (undirected, without loops or multiple edges) with n edges: we say that a collection S of edges forms a *Hamiltonian cycle* in G if the set of edges in S is connected and every vertex of G belongs to exactly two edges from S, see Fig. 1.1.

A graph with m vertices may contain as many as $\frac{(m-1)!}{2}$ different Hamiltonian cycles and it is believed (known, if $\mathbf{P} \neq \mathbf{NP}$) that it is computationally hard to find at least one for a graph G supplied by a clever adversary.

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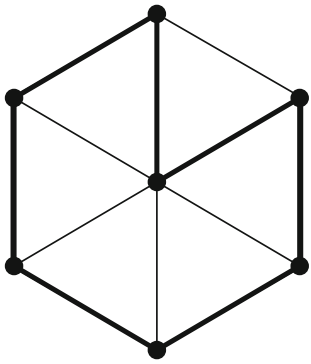

Fig. 1.1 A graph with 7 vertices, 12 edges and a Hamiltonian cycle (*thick lines*)

Sometimes we allow $\mathcal{F}$ to be a family of multisets, in which case we replace

$$\prod_{i \in S} x_i \quad \longrightarrow \quad \prod_{i \in S} x_i^{\mu_i}$$

in formula (1.1), where μ_i is the multiplicity of i in S.

Sometimes we know $p_{\mathcal{F}}$ perfectly well even if we are unable to write it explicitly as a sum of monomials due to the lack of time. For example, if $\mathcal{F} = 2^{\{1,\ldots,n\}}$ is the set of all subsets, we have

$$p_{2^{\{1,\ldots,n\}}}(x_1, \ldots, x_n) = \sum_{S \subset \{1,\ldots,n\}} \prod_{i \in S} x_i = \prod_{i=1}^{n} (1 + x_i) \tag{1.2}$$

and it is hard to argue that we can know $p_{2^{\{1,\ldots,n\}}}$ any better than by the succinct product in (1.2). Our experience teaches us, however, that the cases like (1.2) are quite rare. For some mysterious reasons they all seem to reduce eventually to some determinant enumerating perfect matchings in a planar graph, see [Ba82], [Va08] and Chap. 10 of [Ai07] for examples and recall that a *perfect matching* in a graph is a collection of edges that contains every vertex of the graph exactly once (see Fig. 4.1) and that the graph is *planar* if it can be drawn in the plane so that no two edges can possibly intersect in a point other than their common vertex (see Fig. 4.8).

Although in Sect. 4.3 of the book we describe the classical Kasteleyn's construction expressing the partition function of perfect matchings in a planar graph as a determinant (more precisely, as a Pfaffian), the focus of the book is different. Since the efficient exact computation of $p_{\mathcal{F}}$ in most interesting situations is believed to be impossible (unless the computational complexity hierarchy collapses, that is, unless $\mathbf{P} = \#\mathbf{P}$), we are interested in situations when $p_{\mathcal{F}}$ can be efficiently approximated. By *efficiently approximated* we understand that we can compute $p_{\mathcal{F}}$ approximately for all $x = (x_1, \ldots, x_n)$ in some sufficiently interesting domain, but not only. We also approximate $p_{\mathcal{F}}$ by some "nice function", whose behavior we understand reasonably well. We concentrate mostly on the following three approaches.

Scaling. It may happen that there is a sufficiently rich group of transformations, for example of the type $x_i \longmapsto \lambda_i x_i$ for some λ_i, which change the value of the polynomial $p_{\mathcal{F}}(x_1, \ldots, x_n)$ in some obvious way and such that after factoring that group out, we are left with a function that varies little. This is the case for the permanent (Sect. 3.5), hafnian (Sect. 4.2) and their higher-dimensional extensions (Sects. 4.4 and 4.5). A closely related approach expresses $p_{\mathcal{F}}$ as the coefficient of a monomial $y_1^{\alpha_1} \cdots y_N^{\alpha_N}$ in some explicit polynomial $P(y_1, \ldots, y_N)$ and obtains an estimate of $p_{\mathcal{F}}$ via solution of a convex optimization problem of minimizing $y_1^{-\alpha_1} \cdots y_N^{-\alpha_N} P(y_1, \ldots, y_N)$ for $y_1, \ldots, y_N > 0$. We apply this approach to estimate partition functions of flows (Chap. 8).

Correlation decay. We choose a variable (or a small set of variables), say x_n, and define $p_{\mathcal{F}_n}$ as the sum of the monomials of $p_{\mathcal{F}}$ containing x_n. It may happen that there is some metric on the set $\{x_1, \ldots, x_n\}$ of variables such that the ratio $p_{\mathcal{F}_n}(x_1, \ldots, x_n) / p_{\mathcal{F}}(x_1, \ldots, x_n)$ does not depend much on the variables x_i that are sufficiently far away from x_n in that metric. This allows us to fix values of those remote variables to our convenience and quickly approximate the ratio. We then recover $p_{\mathcal{F}}$ by iterating this procedure and telescoping. As a result, we approximate $\ln p_{\mathcal{F}}(x_1, \ldots, x_n)$ by a sum of functions, each of which depends on a small number of coordinates. We apply this method to the matching polynomial (Sect. 5.2) and to the independence polynomial of a graph (Sects. 6.3 and 6.4).

Interpolation. Suppose that the polynomial $p_{\mathcal{F}}$ has no zeros in a domain $\Omega \subset \mathbb{C}^n$. It turns out that $\ln p_{\mathcal{F}}$ is well approximated in a slightly smaller domain $\Omega' \subset \Omega$ by a low degree Taylor polynomial, sometimes after a change of coordinates (Sect. 2.2). We demonstrate this approach for the permanent (Sects. 3.6 and 3.7) and hafnian (Sect. 4.1), their higher-dimensional extensions (Sect. 4.4), for the matching polynomial (Sect. 5.1) and the independence polynomial of a graph (Sect. 6.1), and for the graph homomorphism partition function (Chap. 7). In our opinion, this is the most general approach.

The correlation decay approach appears to be closely related to a probabilistic approach, known as the Markov Chain Monte Carlo method. Assuming that $x_1 > 0, \ldots, x_n > 0$, we consider the family $\mathcal{F}$ as a finite probability space, with

$$\mathbf{Pr}(S) = \left(\prod_{i \in S} x_i \right) / p_{\mathcal{F}}(x_1, \ldots, x_n) \quad \text{for} \quad S \in \mathcal{F}. \tag{1.3}$$

Suppose that we can sample a random set $S \in \mathcal{F}$ in accordance with the probability distribution (1.3). Then we can measure the frequency of how often a random S contains a particular element of the ground set, say n, and hence we can estimate the ratio $p_{\mathcal{F}_n}(x_1, \ldots, x_n) / p_{\mathcal{F}}(x_1, \ldots, x_n)$, which is also the goal of the correlation decay method. To sample a random $S \in \mathcal{F}$, we perform a random walk on $\mathcal{F}$ by starting with some particular S and, at each step, trying to modify $S \longmapsto \widehat{S}$ by a random move of the type $\widehat{S} := (S \setminus I) \cup J$ for some small sets $I, J \subset \{1, \ldots, n\}$ performed with probability proportional to

$$\frac{\mathbf{Pr}\,(\widehat{S})}{\mathbf{Pr}\,(S)} = \left(\prod_{j\in J} x_j\right)\left(\prod_{i\in I} x_i^{-1}\right).$$

It stands to reason that if the ratios of the type $p_{\mathcal{F}_n}(x_1,\ldots,x_n)\,/\,p_{\mathcal{F}}(x_1,\ldots,x_n)$ depend effectively only on a small set of variables, then we can expect the resulting walk to mix rapidly, that is, we should hit more or less random S after performing a moderate number of moves.

The Markov Chain Monte Carlo method resulted in a number of remarkable successes, most notably in a randomized polynomial time approximation algorithm for the permanent of a non-negative matrix [J+04]. However, we do not discuss it in this book. First, there are excellent books such as [Je03] describing the method in detail and second, we are interested in analytic properties of partition functions that make them amenable to computation (approximation). Granted, the fact that randomized algorithms are often very efficient must be telling us something important about analytic properties of the functions they approximate, but at the moment we hesitate to say what exactly.

Why this is interesting. Why do we care to approximate $p_{\mathcal{F}}$ in (1.1)?

For one thing, it gives us some information about complicated combinatorial families. As an example, let us consider the family $\mathcal{H}$ of all Hamiltonian cycles in a complete graph K_m (undirected, without loops or multiple edges) with m vertices $1,\ldots,m$. Hence to every edge (i,j) of K_m we assign a variable x_{ij}, to every Hamiltonian cycle in K_m we assign a monomial that is the product of the variables x_{ij} on the edges of the cycle, and we define $p_{\mathcal{H}}$ by summing up all monomials attached to the Hamiltonian cycles in K_m. If we let $x_{ij}=1$ for all edges (i,j) then the value of $p_{\mathcal{H}}$ is just the number of Hamiltonian cycles in K_m, which is $(m-1)!/2$. If we assign $x_{ij}=1$ for some edges of K_m and $x_{ij}=0$ for all other edges of K_m, then the value of $p_{\mathcal{H}}$ is the number of Hamiltonian cycles in the graph G consisting of the edges selected by the condition $x_{ij}=1$ (generally, it is computationally hard even to tell $p_{\mathcal{H}}$ from 0).

Looking at the problem of counting Hamiltonian cycles through the prism of the partition function $p_{\mathcal{H}}$ allows us to interpolate between a trivial problem (counting Hamiltonian cycles in the complete graph) and an impossible one (counting Hamiltonian cycles in an arbitrary graph) and find some middle ground. Given a graph G with vertices $1,\ldots,m$, let us fix a small $\epsilon>0$ (think $\epsilon=10^{-10}$) and let us define

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is an edge of } G \\ \epsilon & \text{otherwise.} \end{cases}$$

In this case, $p_{\mathcal{H}}$ still enumerates Hamiltonian cycles in the complete graph K_m, but it does so deliberately. It counts every Hamiltonian cycle in G with weight 1, while every Hamiltonian cycle in K_m that contains r non-edges of G is counted with weight ϵ^r. In Sect. 3.8, we show that it is quite easy to approximate $p_{\mathcal{H}}$ within a factor of $m^{O(\ln m)}$, where the implicit constant in the "O" notation depends on ϵ. This gives us

some idea about Hamiltonian cycles in G: for example, we can separate graphs G with many Hamiltonian cycles (the value of $p_{\mathcal{H}}$ is large) from graphs G that do not acquire a single Hamiltonian cycle unless sufficiently many new edges are added to G (the value of $p_{\mathcal{H}}$ is small).

Two particular topics discussed in this book are

(1) connections between the computational complexity of partition functions and their complex zeros

and

(2) connections between computational complexity and "phase transition" in physics.

In statistical physics, one deals with the probability space $\mathcal{F}$ defined by (1.3) (sets $S \in \mathcal{F}$ are called "configurations"), where $x_i = e^{\beta_i/t}$ for some constants $\beta_i > 0$ and a real parameter t, interpreted as temperature. As the ground set $\{1, \ldots, n\}$ and the set $\mathcal{F}$ of configurations grow in some regular way, one can consider two related, though not identical notions of phase transition. The first notion has to do with a complex zero of $p_{\mathcal{F}}$, as a function of t, approaching the positive real axis at some "critical temperature" $t_c > 0$. This implies the loss of smoothness or even continuity for various physically meaningful quantities, expressed in terms of $\ln p_{\mathcal{F}}$ and its derivatives [YL52]. The second notion of phase transition has to do with the appearance or disappearance of "long-range correlations". Typically, at a high temperature t (that is, when x_i are close to 1), there is no long-range correlation: the probability that S contains a given element i of the ground set is not affected by whether S contains another element j, far away from i in some natural metric. As the temperature t falls (and hence x_i grow), such a dependence may appear. These two notions of phase transition are related though apparently not identical, see [DS87] and [Ci87], we discuss this when we talk about the Ising model in Sect. 7.4 .

The correlation decay approach emphasizing (2) was introduced by Bandyopadhyay and Gamarnik [BG08] and independently by Weitz [We06] and is generally well-known in the computational community, while (1) is relatively less articulated but appears to be no less interesting. Curiously, while the first type of phase transition is associated with complex zeros of the partition function approaching the positive real axis, as far as our ability to approximate is concerned, a priori this does not represent an insurmountable obstacle. What hinders our ability to compute are the complex zeros "blocking" the reference point in the vicinity of which $p_{\mathcal{F}}$ looks easy, such as the point $x_{ij} = 1$ for the partition function $p_{\mathcal{H}}$ of Hamiltonian cycles, see also our discussion in Sect. 2.2. The ways of statistical physics and those of computational complexity diverge at this point, which is probably explained by the fact that the temperature in the physical world is necessarily a real number, while for computational purposes we can manipulate with a complex temperature just as easily.

We stick to the language of combinatorics but the objects and phenomena discussed in this book have also their names in physics. Thus the "matching polynomial" of Chap. 5 corresponds to the "monomer-dimer model", the "graph homomorphism partition function" in Chap. 7 corresponds to a "spin system", while the cut partition

function of Sect. 7.4 corresponds to a "ferromagnetic spin system". Some of our results, such as in Sects. 3.6, 3.7, 3.8, 4.2, 4.4, 7.1 and 7.2 correspond to the "mean field theory" approach, while some others, such as in Chaps. 5 and 6 correspond to the "hard core" model. For still others, such as in Sects. 3.4, 3.5 and Chap. 8, we were unable to think of an appropriate physics name (though "renormalization" may work for those in Sects. 3.4 an 3.5). We talk about physical implications of results in Sect. 7.4 while discussing the Ising model, which connects several directions explored this book: zeros of partition functions, phase transition, correlation decay, graph homomorphisms and enumeration of perfect matchings.

Finally, this book may be interesting because it contains an exposition of quite recent breakthroughs (available before, to the best of our knowledge, only as preprints, journal or conference proceedings papers). These include the Gurvits approach connecting certain combinatorial quantities with stable polynomials (Sects. 3.3 and 8.1), Csikvári and Lelarge approach to the Bethe-approximation of the permanent (Sects. 5.3 and 5.4) and Weitz correlation decay method for the independence polynomial (Sect. 6.4).

Prerequisites, contents, notation, and assorted remarks. We use some concepts of combinatorics, but only very basic, such as graphs and hypergraphs. All other terms, also very basic, such as matchings, perfect matchings and colorings are explained in the text. We also employ some computational complexity concepts. As we are interested in establishing that some functions can be efficiently computed (approximated), and not in proving that some functions are hard to approximate, we use only some very basic complexity concepts, such as polynomial time algorithm, etc. The book [PS98] will supply more than enough prerequisites in combinatorics and computational complexity (but see also more recent and comprehensive [AB09] and [Go08]). We also require modest amounts of linear algebra, real and complex analysis. This book should be accessible to an advanced undergraduate.

In Chap. 2, we develop our toolbox. First, we discuss various topics in convexity: convex and concave functions, entropy and Bethe-entropy, Gauss-Lucas theorem on the zeros of the derivative of a complex polynomial, the capacity of real polynomials and the Prékopa-Leindler inequality. Then we present one of our main tools, interpolation, which allows us to approximate the logarithm of a multivariate polynomial p by a low degree polynomial in a domain, given that there are no complex zeros of p in a slightly larger domain. We discuss interlacing polynomials, $\mathbb{H}$-stable polynomials (polynomials with no roots in the open upper half-plane of $\mathbb{C}$) and $\mathbb{D}$-stable polynomials (polynomials with no roots in the closed unit disc in $\mathbb{C}$).

Then we begin our study of partition functions in earnest.

In Chap. 3, we start slowly with the permanent, as it is very easy to define and it has a surprisingly rich structure. All this makes the permanent a very natural candidate to try our toolbox on.

In Chap. 4, we consider extensions of the permanent to non-bipartite graphs (hafnians) and hypergraphs (multi-dimensional permanents). We also consider the mixed discriminant, which is a generalization of the permanent and of the determinant

simultaneously. We observe that some properties of the permanent can be extended to those more general objects, while some other cannot.

In Chap. 5, we consider the matching polynomial of a graph, a relative of the permanent and hafnian. Here we introduce the correlation decay method, which, as Bayati, Gamarnik, Katz, Nair and Tetali showed [B+07], looks particularly elegant and simple in the case of the matching polynomial. It turns out to be very useful too and provides some additional insight into the permanent.

In Chap. 6, we discuss the independence polynomial of a graph. We prove Dobrushin's bound on the complex roots and also present the correlation decay approach at its most technical. We discuss an open question due to Sokal [S01b], which, if answered affirmatively, would allow us to bridge the gap between different degrees of approximability afforded by the interpolation and by correlation decay approaches.

In Chap. 7, we present combinatorial partition functions at their most general. Here we rely entirely on our interpolation technique, although some of the results can be obtained by the correlation decay approach [LY13]. We also prove the Circle Theorem of Lee and Yang and discuss the Ising model in some detail.

In Chap. 8, we consider partition functions associated with multisets. We study the partition functions of 0-1 and non-negative integer flows, which present yet another extension of permanents. Permanents also supply our main technical tool.

Sections, theorems, lemmas, and formulas are numbered separately inside each chapter. Figures are numbered consecutively in each chapter. For example, Fig. 4.3 is the third figure in Chap. 4.

We use $\Re$ to denote the real part of a complex number and $\Im$ to denote the imaginary part of a complex number, so that $\Re z = a$ and $\Im z = b$ for $z = a + ib$. We denote by $|X|$ the cardinality of a finite set X.

Finally, the product of complex numbers from an empty set is always 1.

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Chapter 2
Preliminaries

We assemble our toolbox from real and complex analysis. The main topics are inequalities inspired by convexity, polynomials with no roots in a particular domain and relations between convexity and restrictions on the location of the roots. We discuss the entropy of partitions, the Bethe-entropy, the Prékopa–Leindler inequality for integrals and the capacity of polynomials with non-negative real coefficients as a way to estimate a particular coefficient of a multivariate polynomial by solving a convex optimization problem. We discuss polynomials with real roots, polynomials with no roots in the open upper half-plane ($\mathbb{H}$-stable polynomials) and polynomials with no roots in the closed unit disc ($\mathbb{D}$-stable polynomials). We prove the Gauss–Lucas Theorem for the location of the roots of the derivative of a polynomial, the Gurvits Theorem on the capacity of $\mathbb{H}$-stable polynomials and establish log-concavity of the coefficients of real-rooted polynomials. We introduce the Taylor polynomial interpolation method, which allows us to obtain computationally efficient low-degree approximations of a polynomial in a complex domain, provided the polynomial has no zeros in a slightly larger domain.

2.1 Convexity

2.1.1 Convex functions. In what follows, some convex/concave functions will play an important role. A set $A \subset \mathbb{R}^d$ is called *convex* provided

$$\alpha x + (1-\alpha)y \in A \quad \text{for all} \quad x, y \in A \quad \text{and all} \quad 0 \le \alpha \le 1.$$

It follows then that

$$\sum_{i=1}^{n} \alpha_i x_i \in A \quad \text{provided} \quad x_i \in A, \quad \alpha_i \ge 0 \quad \text{for} \quad i = 1, \dots, n$$
$$\text{and} \quad \sum_{i=1}^{n} \alpha_i = 1.$$

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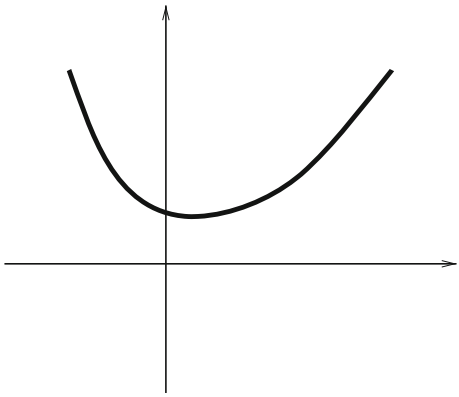

Fig. 2.1 The graph of a convex function

Let $A \subset \mathbb{R}^d$ be a convex set. A function $f : A \longrightarrow \mathbb{R}$ is called *convex* provided

$$f\big(\alpha x+(1-\alpha)y\big) \ \leq \ \alpha f(x)+(1-\alpha)f(y) \quad \text{for all} \quad x, y \in A \quad \text{and all} \quad 0 \leq \alpha \leq 1,$$

see Fig. 2.1

The function f is called *strictly convex* if the above inequality is strict whenever $x \neq y$ and $0 < \alpha < 1$. It is easy to show that if f is convex then

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i f(x_i) \quad \text{provided} \quad x_i \in A, \quad \alpha_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n$$
$$\text{and} \quad \sum_{i=1}^{n} \alpha_i = 1.$$

A function $f : A \longrightarrow \mathbb{R}$ is called *concave* provided

$$f\big(\alpha x+(1-\alpha)y\big) \ \geq \ \alpha f(x)+(1-\alpha)f(y) \quad \text{for all} \quad x, y \in A \quad \text{and all} \quad 0 \leq \alpha \leq 1,$$

see Fig. 2.2.

The function f is called *strictly concave* if the above inequality is strict whenever $x \neq y$ and $0 < \alpha < 1$. It is easy to show that if f is concave then

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \geq \sum_{i=1}^{n} \alpha_i f(x_i) \quad \text{provided} \quad x_i \in A, \quad \alpha_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n$$
$$\text{and} \quad \sum_{i=1}^{n} \alpha_i = 1.$$

Here are some functions whose convexity/concavity we will repeatedly use.

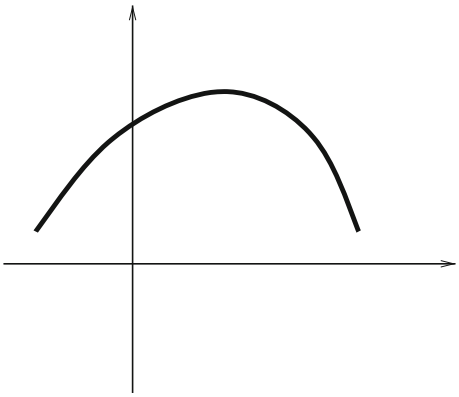

Fig. 2.2 The graph of a concave function

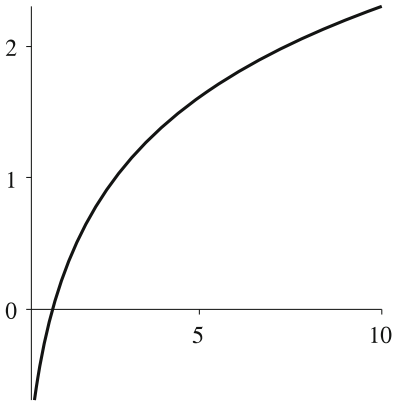

Fig. 2.3 The graph of $\ln x$

2.1.1.1 Logarithm. As is well known, the function

$$f(x) = \ln x \quad \text{for} \quad x > 0$$

is strictly concave, see Fig. 2.3.

In particular,

$$\ln\left(\sum_{i=1}^{n} \alpha_i x_i\right) \geq \sum_{i=1}^{n} \alpha_i \ln x_i \quad \text{provided} \quad x_i, \alpha_i > 0 \quad \text{for} \quad i = 1, \ldots, n$$
$$\text{and} \quad \sum_{i=1}^{n} \alpha_i = 1.$$

Exponentiating, we obtain the *arithmetic-geometric mean inequality*:

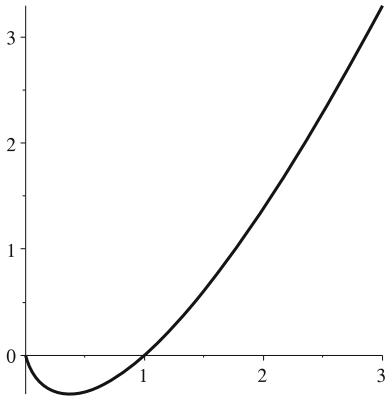

Fig. 2.4 The graph of $x \ln x$

$$\sum_{i=1}^{n} \alpha_i x_i \geq \prod_{i=1}^{n} x_i^{\alpha_i} \quad \text{provided} \quad x_i, \alpha_i > 0 \quad \text{for} \quad i = 1, \ldots, n$$

$$\text{and} \quad \sum_{i=1}^{n} \alpha_i = 1.$$

2.1.1.2 The function $f(x) = x \ln x$. It is easy to check that the function

$$f(x) = x \ln x \quad \text{for} \quad x > 0$$

is strictly convex, see Fig. 2.4, and, consequently, the function

$$h(x) = x \ln \frac{1}{x} \quad \text{for} \quad x > 0$$

is strictly concave.

2.1.1.3 Exponential substitution. Let $p(x_1, \ldots, x_n)$ be a polynomial with non-negative real coefficients. Then the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$f(t_1, \ldots, t_n) = \ln p\left(e^{t_1}, \ldots, e^{t_n}\right)$$

is convex. Indeed, it suffices to check that the restriction h of f onto every line in $\mathbb{R}^n$ is convex. Such a restriction h looks as

$$h(t) = \ln\left(\sum_{i=1}^{m} \alpha_i e^{\lambda_i t}\right),$$

where $\lambda_1, \ldots, \lambda_m$ are real and $\alpha_1, \ldots, \alpha_m$ are positive real. It suffices then to check that $h''(t) \geq 0$ for all $t \in \mathbb{R}$. Denoting

$$g(t) = \sum_{i=1}^{m} \alpha_i e^{\lambda_i t},$$

we obtain

$$h'(t) = \frac{g'(t)}{g(t)} \quad \text{and} \quad h''(t) = \frac{g''(t)g(t) - g'(t)g'(t)}{g^2(t)}$$

where

$$\begin{aligned} g''(t)g(t) - g'(t)g'(t) &= \sum_{i,j=1}^{m} \lambda_i^2 \alpha_i \alpha_j e^{(\lambda_i+\lambda_j)t} - \sum_{i,j=1}^{m} \lambda_i \lambda_j \alpha_i \alpha_j e^{(\lambda_i+\lambda_j)t} \\ &= \sum_{\substack{\{i,j\} \\ i \neq j}} \left(\lambda_i^2 + \lambda_j^2 - 2\lambda_i \lambda_j\right) \alpha_i \alpha_j e^{(\lambda_i+\lambda_j)t} \\ &= \sum_{\substack{\{i,j\} \\ i \neq j}} \left(\lambda_i - \lambda_j\right)^2 \alpha_i \alpha_j e^{(\lambda_i+\lambda_j)t} \geq 0. \end{aligned}$$

2.1.2 Entropy. Let us consider the simplex $\Delta_n \subset \mathbb{R}^n$ consisting of all vectors $x = (\xi_1, \ldots, \xi_n)$ such that $\xi_i \geq 0$ for $i = 1, \ldots, n$ and $\xi_1 + \ldots + \xi_n = 1$. For $x \in \Delta_n$ we define the *entropy* H by

$$H(x) = \sum_{i=1}^{n} \xi_i \ln \frac{1}{\xi_i} \quad \text{where} \quad x = (\xi_1, \ldots, \xi_n)$$

and the corresponding term is 0 if $\xi_i = 0$. It follows from Sect. 2.1.1.2 that H is strictly concave. Therefore H attains its minimum value on Δ_n at an extreme point of Δ_n, that is, where $\xi_i = 1$ for some i and $\xi_j = 0$ for all $j \neq i$. In particular,

$$H(x) \geq 0 \quad \text{for all} \quad x \in \Delta_n.$$

Clearly, H is a symmetric function of $\xi_1, \ldots, \xi_n$, so the value of H depends on the multiset $\{\xi_1, \ldots, \xi_n\}$ but not on the order of ξ_is.

By the concavity and symmetry of H, the largest value of H on Δ_n is attained when $\xi_1 = \ldots = \xi_n = 1/n$, so

$$H(x) \leq \ln n \quad \text{for all} \quad x \in \Delta_n. \tag{2.1.2.1}$$

A multiset of non-negative numbers summing up to 1 is naturally interpreted as a probability distribution. Let Ω be a probability space and let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be its partition into finitely many pairwise disjoint events $F_1, \ldots, F_n$, so that

$$\Omega = \bigcup_{F_i \in \mathcal{F}} F_i \quad \text{and} \quad F_i \cap F_j = \emptyset \quad \text{for} \quad i \neq j.$$

We define the *entropy of the partition* $\mathcal{F}$ by

$$H(\mathcal{F}) = H(\{\mathbf{Pr}\, F_i\}) = \sum_{i=1}^{n} p_i \ln \frac{1}{p_i} \quad \text{where} \quad p_i = \mathbf{Pr}(F_i).$$

In particular, by (2.1.2.1),

$$H(\mathcal{F}) \leq \ln n \quad \text{provided } \mathcal{F} \text{ consists of not more than } n \text{ events.} \tag{2.1.2.2}$$

We say that a finite partition $\mathcal{G}$ *refines* a partition $\mathcal{F}$ if every event in the partition $\mathcal{G}$ lies in some event in the partition $\mathcal{F}$, see Fig. 2.5, in which case we write $\mathcal{F} \preceq \mathcal{G}$. We often call events of a coarser partition *blocks*.

Given a pair of partitions $\mathcal{F} \preceq \mathcal{G}$, we define the *conditional entropy* of $\mathcal{G}$ with respect to $\mathcal{F}$ as follows:

$$H(\mathcal{G}|\mathcal{F}) = \sum_{F \in \mathcal{F}} \mathbf{Pr}(F) \left(\sum_{\substack{G \in \mathcal{G} \\ G \subset F}} \frac{\mathbf{Pr}(G)}{\mathbf{Pr}(F)} \ln \frac{\mathbf{Pr}(F)}{\mathbf{Pr}(G)} \right)$$

(if $\mathbf{Pr}(F) = 0$ for some F, the corresponding term in the sum is 0). In words: each event F of the partition $\mathcal{F}$ such that $\mathbf{Pr}(F) > 0$ we consider as a probability space endowed with the conditional probability measure, compute the entropy of the partition of F by events of $\mathcal{G}$ and average that entropy over all events $F \in \mathcal{F}$.

For $\omega \in \Omega$, let us denote by $F(\omega)$ the event of $\mathcal{F}$ containing ω, considered as a probability space as before, and let $\mathcal{F}(\omega)$ be the partition of $F(\omega)$ induced by $\mathcal{G}$. Assuming that Ω is finite, we can write

$$H(\mathcal{G}|\mathcal{F}) = \sum_{\omega \in \Omega} \mathbf{Pr}(\omega) H(\mathcal{F}(\omega)). \tag{2.1.2.3}$$

It is not hard to check that

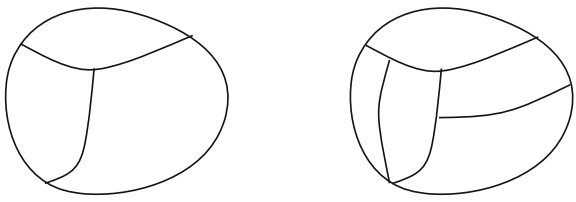

Fig. 2.5 A partition and its refinement

$$H(\mathcal{G}) = H(\mathcal{F}) + H(\mathcal{G}|\mathcal{F}).$$

Moreover, if $\mathcal{F}_1 \preceq \mathcal{F}_2 \preceq \ldots \preceq \mathcal{F}_m$, iterating the above identity, we get

$$H(\mathcal{F}_m) = H(\mathcal{F}_1) + \sum_{i=1}^{m-1} H(\mathcal{F}_{i+1}|\mathcal{F}_i) \tag{2.1.2.4}$$

see, for example, [Kh57].

2.1.3 Bethe-entropy. Let $\Delta_n \subset \mathbb{R}^n$ be, as above, the simplex of all n-vectors $x = (\xi_1, \ldots, \xi_n)$ such that $\xi_i \geq 0$ for $i = 1, \ldots, n$ and $\xi_1 + \ldots + \xi_n = 1$. We assume that $n \geq 2$ and for $x \in \Delta_n$, we define

$$g(x) = \sum_{i=1}^{n} \left(\xi_i \ln \frac{1}{\xi_i} + (1 - \xi_i) \ln(1 - \xi_i) \right) \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n)\,.$$

We call this function the *Bethe-entropy*. We claim that $g(x)$ is a non-negative concave function on Δ_n. We follow [Gu11], see also [Vo13].

Let

$$\phi(\xi) = \xi \ln \frac{1}{\xi} + (1 - \xi) \ln(1 - \xi),$$

see Fig. 2.6.

We have

$$\phi''(\xi) = \frac{2\xi - 1}{\xi(1 - \xi)},$$

from which ϕ is concave for $0 \leq \xi \leq 1/2$. Since $\phi(0) = \phi(1/2) = 1$, it follows that $\phi(\xi) \geq 0$ for $0 \leq \xi \leq 1/2$.

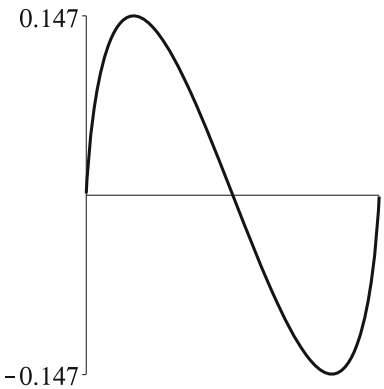

Fig. 2.6 The graph of $\phi(x)$

Hence $g(x) \geq 0$ if $\xi_i \leq 1/2$ for $i = 1, \ldots, n$. Otherwise, there is at most one value of ξ_i, say ξ_n, such that $\xi_n > 1/2$. Therefore, the minimum of the concave function $\sum_{i=1}^{n-1} \phi(\xi_i)$ on the simplex defined by the equation $\xi_1 + \ldots + \xi_{n-1} = 1 - \xi_n$ and inequalities $\xi_i \geq 0$ for $i = 1, \ldots, n-1$ is attained at an extreme point, where all but one ξ_i, say ξ_1, are equal to 0 and $\xi_1 = 1 - \xi_n$. Therefore,

$$g(x) \;\geq\; \phi(\xi_n) + \phi\,(1 - \xi_n) = 0,$$

and hence $g(x)$ is indeed non-negative on Δ_n.

To prove that $g(x)$ is concave, it suffices to prove that the restriction of g onto the relative interior of Δ_n is concave, so we assume that $\xi_1, \ldots, \xi_n > 0$. Computing the Hessian of g at $x = (\xi_1, \ldots, \xi_n)$, we obtain the $n \times n$ diagonal matrix

$$D = \operatorname{diag}\left(\frac{2\xi_1 - 1}{\xi_1(1-\xi_1)}, \ldots, \frac{2\xi_n - 1}{\xi_n(1-\xi_n)}\right).$$

Our goal is to prove that the restriction of the quadratic form with matrix D onto the tangent space to $x \in \Delta_n$ is negative semi-definite, that is

$$\sum_{i=1}^{n} \frac{2\xi_i - 1}{\xi_i(1-\xi_i)}\alpha_i^2 \;\leq\; 0 \quad \text{provided} \quad \sum_{i=1}^{n} \alpha_i = 0. \tag{2.1.3.1}$$

If $\xi_i \leq 1/2$ for all $i = 1, \ldots, n$ then (2.1.3.1) obviously holds. Otherwise, there is at most one coordinate ξ_i, say, ξ_n, such that $\xi_n > 1/2$. If $\alpha_n = 0$ then (2.1.3.1) holds, so we can assume that $\alpha_n \neq 0$. Scaling, if necessary, we can further assume that $\alpha_n = -1$.

Let us denote

$$\beta_i = \frac{2\xi_i - 1}{\xi_i(1-\xi_i)} \quad \text{for} \quad i = 1, \ldots, n-1.$$

The maximum value of the negative definite quadratic form

$$(\alpha_1 + \ldots + \alpha_{n-1}) \longmapsto \sum_{i=1}^{n-1} \beta_i \alpha_i^2$$

on the affine subspace defined by the equation

$$\alpha_1 + \ldots + \alpha_{n-1} = 1$$

is attained at

$$\alpha_i = \frac{\lambda}{\beta_i} \quad \text{for} \quad i = 1, \ldots, n-1 \quad \text{and} \quad \lambda = \left(\sum_{i=1}^{n-1} \frac{1}{\beta_i}\right)^{-1}$$

and hence is equal to

$$\lambda = \left(\sum_{i=1}^{n-1} \frac{1}{\beta_i} \right)^{-1} = \left(\sum_{i=1}^{n-1} \frac{\xi_i(1-\xi_i)}{2\xi_i - 1} \right)^{-1}.$$

Consequently,

$$\sum_{i=1}^{n} \frac{2\xi_i - 1}{\xi_i(1-\xi_i)} \alpha_i^2 \leq \frac{2\xi_n - 1}{\xi_n(1-\xi_n)} + \left(\sum_{i=1}^{n-1} \frac{\xi_i(1-\xi_i)}{2\xi_i - 1} \right)^{-1}$$
$$\text{provided} \quad \sum_{i=1}^{n-1} \alpha_i = 1 \quad \text{and} \quad \alpha_n = -1. \tag{2.1.3.2}$$

On the other hand, the function

$$\xi \longmapsto \frac{\xi(1-\xi)}{2\xi - 1} \quad \text{for} \quad 0 \leq \xi < \frac{1}{2}$$

is concave, as we have

$$\frac{d^2}{d\xi^2} \frac{\xi(1-\xi)}{2\xi - 1} = \frac{2}{(2\xi - 1)^3} < 0 \quad \text{provided} \quad 0 \leq \xi < \frac{1}{2}.$$

Consequently, the minimum value of the concave function

$$(\xi_1, \ldots, \xi_{n-1}) \longmapsto \sum_{i=1}^{n-1} \frac{\xi_i(1-\xi_i)}{2\xi_i - 1}$$

on the simplex

$$\xi_1 + \ldots + \xi_{n-1} = 1 - \xi_n \quad \text{and} \quad \xi_1, \ldots, \xi_{n-1} \geq 0$$

is attained at an extreme point, where all but one ξ_i are equal to 0 and the remaining value of ξ_i is $1 - \xi_n$. Then from (2.1.3.2) we conclude that

$$\sum_{i=1}^{n} \frac{2\xi_i - 1}{\xi_i(1-\xi_i)} \alpha_i^2 \leq \frac{2\xi_n - 1}{\xi_n(1-\xi_n)} + \frac{1 - 2\xi_n}{\xi_n(1-\xi_n)} = 0$$
$$\text{provided} \quad \sum_{i=1}^{n-1} \alpha_i = 1 \quad \text{and} \quad \alpha_n = -1,$$

and (2.1.3.1) follows, establishing the concavity of g.

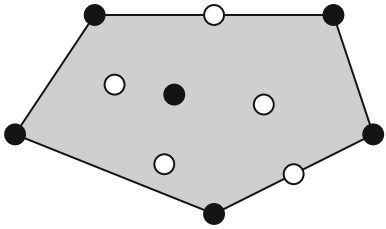

Fig. 2.7 The roots of f (*black dots*) and the roots of f' (*white dots*)

2.1.4 Gauss–Lucas Theorem. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a non-constant polynomial. The Gauss–Lucas Theorem states that the roots of $f'(z)$ lie in the convex hull of the roots of f, see Fig. 2.7.

Indeed, without loss of generality, we assume that f is monic. Let $\gamma_1, \ldots, \gamma_n$ be the roots of f, counted with multiplicities, so

$$f(z) = \prod_{k=1}^{n} (z - \gamma_k).$$

Let β be a root of f', so

$$0 = f'(\beta) = \sum_{k=1}^{n} \prod_{m \neq k} (\beta - \gamma_m) \quad \text{and hence} \quad \sum_{k=1}^{n} \prod_{m \neq k} \overline{(\beta - \gamma_m)} = 0.$$

If $\beta = \gamma_j$ for some j, the result follows instantly. Otherwise, multiplying the last equation by $\prod_{m=1}^{n} (\beta - \gamma_m)$, we obtain

$$\sum_{k=1}^{n} (\beta - \gamma_k) \prod_{m \neq k} |\beta - \gamma_m|^2 = 0.$$

Denoting

$$\alpha_k = \frac{\prod_{m \neq k} |\beta - \gamma_m|^2}{\sum_{k=1}^{m} \prod_{m \neq k} |\beta - \gamma_m|^2},$$

we write β as a convex combination of $\gamma_1, \ldots, \gamma_m$:

$$\beta = \sum_{k=1}^{m} \alpha_k \gamma_k \quad \text{where} \quad \sum_{k=1}^{m} \alpha_k = 1 \quad \text{and} \quad \alpha_k \geq 0 \quad \text{for} \quad k = 1, \ldots, m.$$

2.1.5 Capacity. Let

$$p(x_1, \ldots, x_n) = \sum_{m \in M} a_m \mathbf{x}^m \quad \text{where} \quad \mathbf{x}^m = x_1^{\mu_1} \cdots x_n^{\mu_n} \tag{2.1.5.1}$$

be a polynomial with non-negative real coefficients $a_m \geq 0$ for $m \in M$. Following Gurvits [Gu08, Gu15], given a non-negative integer vector $r = (\rho_1, \ldots, \rho_n)$, we define the *capacity* of p by

$$\mathrm{cap}_r(p) = \inf_{x_1,\ldots,x_n>0} \frac{p(x_1, \ldots, x_n)}{x_1^{\rho_1} \cdots x_n^{\rho_n}}.$$

As follows from Sect. 2.1.1.3, the substitution $x_i = e^{t_i}$ for $i = 1, \ldots, n$ expresses the capacity in terms of the infimum of a convex function on $\mathbb{R}^n$:

$$\ln \mathrm{cap}_r(p) = \inf_{t_1,\ldots,t_n} \ln p\left(e^{t_1}, \ldots, e^{t_n}\right) - \rho_1 t_1 - \ldots - \rho_n t_n. \tag{2.1.5.2}$$

This makes the capacity efficiently computable, see, for example, [Ne04], provided the value of the polynomial p is efficiently computable for any given $x_1, \ldots, x_n$.

It follows from (2.1.5.2) that the function $r \longmapsto \ln \mathrm{cap}_r(p)$, being the point-wise minimum of a family of affine functions, is concave, meaning that if $m_1, \ldots, m_k$ are non-negative integer vectors and

$$r = \sum_{i=1}^{k} \alpha_i m_i \quad \text{where} \quad \sum_{i=1}^{k} \alpha_i = 1 \quad \text{and} \quad \alpha_i \geq 0 \quad \text{for} \quad i = 1, \ldots, k$$

is also a non-negative integer vector, then

$$\ln \mathrm{cap}_r p \;\geq\; \sum_{i=1}^{k} \alpha_i \ln \mathrm{cap}_{m_i} p.$$

We get an immediate upper bound on the coefficients of p in terms of the capacity:

$$a_m \;\leq\; \mathrm{cap}_m(p) \quad \text{for all} \quad m \in M.$$

We obtain a complementary lower bound if we assume that the function $m \longmapsto \ln a_m$ is (approximately) concave. More precisely, we prove the following statement:

Let $0 < \beta \leq 1$ be a real and let $r \in M$ be an index in (2.1.5.1) such that whenever

$$r = \sum_{i=1}^{k} \alpha_i m_i \quad \text{where} \quad m_i \in M, \quad \alpha_i > 0 \quad \text{for} \quad i = 1, \ldots, k$$

$$\text{and} \quad \sum_{i=1}^{m} \alpha_i = 1,$$

we have

$$a_r \;\geq\; \beta \prod_{i=1}^{k} a_{m_i}^{\alpha_i}.$$

Then

$$a_r \;\geq\; \frac{\beta}{|M|}\mathrm{cap}_r(p), \tag{2.1.5.3}$$

where $|M|$ is the number of monomials in the expansion (2.1.5.1).

Without loss of generality, we assume that $a_m > 0$ for all $m \in M$. Let us consider a lifting

$$M \longrightarrow \mathbb{R}^{n+1}, \quad m \longmapsto (m,\ \ln a_m)\,.$$

Let us choose an arbitrary $\gamma > \ln a_r - \ln \beta$ and let us consider a closed ray

$$R = \{(r, \alpha) :\ \alpha \geq \gamma\}\,.$$

Then R does not intersect the convex hull of the points $(m, \ln a_m)$ for $m \in M \setminus \{r\}$. Therefore, there is a linear function separating R from the set $(m, \ln a_m)$ for $m \in M \setminus \{r\}$. Hence writing $r = (\rho_1, \ldots, \rho_n)$, we conclude that there are real $t_1, \ldots, t_n; t_{n+1}$ such that

$$t_{n+1}\alpha + \sum_{i=1}^{n} \rho_i t_i \;>\; t_{n+1} \ln a_m + \sum_{i=1}^{n} \mu_i t_i$$
$$\text{for all} \quad m \in M \setminus \{r\}, \quad m = (\mu_1, \ldots, \mu_n) \quad \text{and all} \quad \alpha \geq \gamma.$$

Moreover, we can choose $t_1, \ldots, t_n; t_{n+1}$ sufficiently generic, so that $t_{n+1} \neq 0$, in which case we must necessarily have $t_{n+1} > 0$ and which we can further scale to $t_{n+1} = 1$, see Fig. 2.8.

Hence we conclude that

$$\gamma + \sum_{i=1}^{n} \rho_i t_i \;>\; \ln a_m + \sum_{i=1}^{n} \mu_i t_i \quad \text{for all} \quad m \in M \setminus \{r\}, \quad m = (\mu_1, \ldots, \mu_n)\,.$$

Since $\gamma > \ln a_r - \ln \beta$ was chosen arbitrarily, we conclude further that

$$\ln a_r + \sum_{i=1}^{n} \rho_i t_i \;\geq\; \ln a_m + \ln \beta + \sum_{i=1}^{n} \mu_i t_i \quad \text{for all} \quad m \in M \setminus \{r\}, \quad m = (\mu_1, \ldots, \mu_n)\,.$$

Letting $x_i = e^{t_i}$ for $i = 1, \ldots, n$, we get

$$a_r x_1^{\rho_1} \cdots x_n^{\rho_n} \;\geq\; \beta a_m x_1^{\mu_1} \cdots x_n^{\mu_n} \quad \text{for all} \quad m \in M \setminus \{r\}, \quad m = (\mu_1, \ldots, \mu_n)\,,$$

from which

$$\beta p\,(x_1, \ldots, x_n) \;\leq\; |M| a_r x_1^{\rho_1} \cdots x^{\rho_n}$$

and (2.1.5.3) follows.

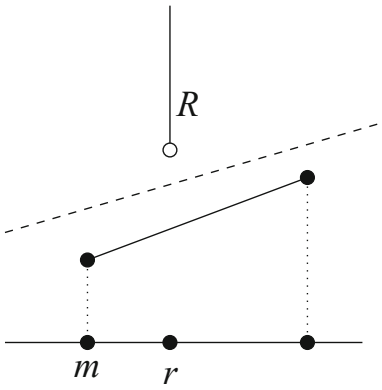

Fig. 2.8 An index m, its lifting, a ray R and an affine hyperplane separating R from the liftings of indices m

2.1.6 Prékopa–Leindler inequality. We will need the following useful inequality. Let $f, f_1, \ldots, f_k : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ be non-negative integrable functions and let $\alpha_1, \ldots, \alpha_k \geq 0$ be reals such that $\alpha_1 + \ldots + \alpha_k = 1$. Suppose further, that

$$f\left(\sum_{i=1}^{k} \alpha_i x_i\right) \geq \prod_{i=1}^{k} f_i^{\alpha_i}(x_i) \quad \text{for all} \quad x_1, \ldots, x_k \in \mathbb{R}^n.$$

Then

$$\int_{\mathbb{R}^n} f(x)\, dx \geq \prod_{i=1}^{k} \left(\int_{\mathbb{R}^n} f_i(x)\, dx\right)^{\alpha_i}.$$

We adapt the proof of Sect. 2.2 of [Le01].

We proceed by induction on the dimension n of the ambient space. The main work is done in dimension 1. For $n = 1$, by continuity we may assume that $f_1, \ldots, f_k$ are strictly positive and continuous. Scaling, if necessary, we may assume further that

$$\int_{-\infty}^{+\infty} f_i(x)\, dx = 1 \quad \text{for} \quad i = 1, \ldots, k.$$

Let us define

$$F_i(t) = \int_{-\infty}^{t} f_i(x)\, dx \quad \text{for} \quad i = 1, \ldots, k.$$

Hence $F_i(t)$ is an increasing function $F_i : \mathbb{R} \longrightarrow (0, 1)$ and let $u_i : (0, 1) \longrightarrow \mathbb{R}$ be its inverse. Thus $u_i(t)$ is also strictly increasing and $F_i(u_i(t)) = t$ for $i = 1, \ldots, k$. We note that F_i and hence u_i are differentiable and that

$$f_i\left(u_i(t)\right) u_i'(t) = 1 \quad \text{for} \quad i = 1, \ldots, k. \tag{2.1.6.1}$$

Let us define

$$u(t) = \sum_{i=1}^{k} \alpha_i u_i(t) \quad \text{for} \quad t \in (0, 1).$$

Making a substitution $x = u(t)$, we get

$$\int_{-\infty}^{+\infty} f(x)\, dx = \int_0^1 f(u(t))u'(t)\, dt = \int_0^1 f\left(\sum_{i=1}^{k} \alpha_i u_i(t)\right)\left(\sum_{i=1}^{k} \alpha_i u_i'(t)\right) dt$$

By the condition of the theorem,

$$f\left(\sum_{i=1}^{k} \alpha_i u_i(t)\right) \geq \prod_{i=1}^{k} f_i^{\alpha_i}\left(u_i(t)\right),$$

while by the arithmetic-geometric mean inequality,

$$\sum_{i=1}^{k} \alpha_i u_i'(t) \geq \prod_{i=1}^{k} \left(u_i'(t)\right)^{\alpha_i}.$$

Summarizing,

$$\int_{-\infty}^{+\infty} f(x)\, dx \geq \int_0^1 \left(\prod_{i=1}^{k} \left(f\left(u_i(t)\right) u_i'(t)\right)^{\alpha_i}\right) dt = 1$$

by (2.1.6.1) and the proof for $n = 1$ follows.

Suppose that $n > 1$. We represent $\mathbb{R}^n = \mathbb{R}^{n-1} \oplus \mathbb{R}$, $x = (y, t)$, and define

$$g(t) = \int_{\mathbb{R}^{n-1}} f(y, t)\, dy \quad \text{and} \quad g_i(t) = \int_{\mathbb{R}^{n-1}} f_i(y, t)\, dy \quad \text{for} \quad i = 1, \ldots, k.$$

Let us choose arbitrary real $t_1, \ldots, t_k$ and let $t = \alpha_1 t_1 + \ldots + \alpha_k t_k$. We define functions $h, h_1, \ldots, h_k : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ by

$$h(y) = f(y, t) \quad \text{and} \quad h_i(y) = f_i(y, t_i) \quad \text{for} \quad i = 1, \ldots, k.$$

Then

$$h\left(\sum_{i=1}^{k} \alpha_i y_i\right) = f\left(\sum_{i=1}^{k} \alpha_i y_i, \sum_{i=1}^{k} \alpha_i t_i\right) \geq \prod_{i=1}^{k} f^{\alpha_i}(y_i, t_i) = \prod_{i=1}^{k} h_i^{\alpha_i}(y_i)$$

and hence by the induction hypothesis

$$g(t) = \int_{\mathbb{R}^{n-1}} h(t)\, dt \;\geq\; \prod_{i=1}^{k} \left(\int_{\mathbb{R}^{n-1}} h_i(y)\, dy \right)^{\alpha_i} = \prod_{i=1}^{k} g_i^{\alpha_i}(t_i).$$

Applying Fubini's Theorem and the inequality in the 1-dimensional case, we get

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_{-\infty}^{+\infty} g(t)\, dt \;\geq\; \prod_{i=1}^{k} \left(\int_{-\infty}^{+\infty} g_i(t)\, dt \right)^{\alpha_i} = \prod_{i=1}^{k} \left(\int_{\mathbb{R}^n} f_i(x)\, dx \right)^{\alpha_i},$$

which completes the induction.

2.2 Polynomial Approximations

We start with a simple lemma.

2.2.1 Lemma. *Let $g(z)$ be a complex polynomial of degree d and let us suppose that*

$$g(z) \neq 0 \quad \textit{for all} \quad |z| \leq \beta,$$

where $\beta > 1$ is a real number. Let us choose a branch of

$$f(z) = \ln g(z) \quad \textit{for} \quad |z| \leq 1$$

and consider its Taylor polynomial

$$p_n(z) = f(0) + \sum_{k=1}^{n} \left(\frac{d^k}{dz^k} f(z) \Big|_{z=0} \right) \frac{z^k}{k!}.$$

Then

$$|f(z) - p_n(z)| \;\leq\; \frac{d}{(n+1)\beta^n(\beta - 1)} \quad \textit{for all} \quad |z| \leq 1.$$

In particular, assuming that $\beta > 1$ is fixed in advance, to achieve

$$|f(1) - p_n(1)| \;<\; \epsilon$$

for some $\epsilon > 0$, it suffices to choose

$$n = O\left(\ln \frac{d}{\epsilon} \right),$$

where the implicit constant in the "O" notation depends only on β.

Proof of Lemma 2.2.1. Let $\alpha_1, \ldots, \alpha_d$ be the roots of $g(z)$, so we can write

$$g(z) = g(0) \prod_{i=1}^{d} \left(1 - \frac{z}{\alpha_i}\right) \quad \text{where} \quad g(0) \neq 0 \quad \text{and} \quad |\alpha_i| > \beta \quad \text{for} \quad i = 1, \ldots, d.$$

Hence

$$f(z) = \ln g(z) = f(0) + \sum_{i=1}^{d} \ln \left(1 - \frac{z}{\alpha_i}\right) \quad \text{for} \quad |z| \leq 1,$$

and expanding the logarithm, we obtain

$$\ln \left(1 - \frac{z}{\alpha_i}\right) = -\sum_{k=1}^{n} \frac{z^k}{k\alpha_i^k} + \xi_{i,n} \quad \text{for} \quad |z| \leq 1$$

where

$$|\xi_{i,n}| = \left| \sum_{k=n+1}^{\infty} \frac{z^k}{k\alpha_i^k} \right| \leq \frac{1}{(n+1)\beta^n(\beta - 1)} \quad \text{for all} \quad |z| \leq 1.$$

Therefore,

$$f(z) = f(0) - \sum_{i=1}^{d} \sum_{k=1}^{n} \frac{z^k}{k\alpha_i^k} + \eta_n \quad \text{for} \quad |z| \leq 1$$

where

$$|\eta_n| \leq \frac{d}{(n+1)\beta^n(\beta - 1)}.$$

To complete the proof, it suffices to notice that

$$\frac{1}{k!} \frac{d^k}{dz^k} f(z)\Big|_{z=0} = -\sum_{i=1}^{d} \frac{1}{k\alpha_i^k}.$$

□

2.2.2 Computing derivatives of $f(z) = \ln g(z)$. Let $f(z) = \ln g(z)$ as in Lemma 2.2.1, where we assume that $g(0) \neq 0$ and hence a branch of $f(z)$ can be chosen in a sufficiently small neighborhood of $z = 0$. Then

$$f'(z) = \frac{g'(z)}{g(z)} \quad \text{and hence} \quad g'(z) = f'(z)g(z).$$

Differentiating the product $k - 1$ times, we obtain

$$\frac{d^k}{dz^k} g(z)\Big|_{z=0} = \sum_{j=0}^{k-1} \binom{k-1}{j} \left(\frac{d^{k-j}}{dz^{k-j}} f(z)\Big|_{z=0}\right) \left(\frac{d^j}{dz^j} g(z)\Big|_{z=0}\right). \tag{2.2.2.1}$$

Combining the Eq. (2.2.2.1) for $k = 1, \ldots, n$, we obtain a triangular system of linear equations in $f^{(k)}(0)$

$$\begin{aligned}
g'(0) &= g(0)f'(0)\\
g''(0) &= g'(0)f'(0) + g(0)f''(0)\\
g^{(3)}(0) &= g''(0)f'(0) + 2g'(0)f''(0) + g(0)f^{(3)}(0)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
g^{(n)}(0) &= g^{(n-1)}(0)f'(0) + (n-1)g^{(n-2)}(0)f''(0) + \ldots + g(0)f^{(n)}(0)
\end{aligned}$$

with coefficients $g(0) \neq 0$ on the diagonal, from which we can compute the derivatives $f^{(k)}(0)$ for $k = 1, \ldots, n$ from $g(0)$ and $g^{(k)}(0)$ for $k = 1, \ldots, n$ in $O(n^2)$ time.

Lemma 2.2.1 allows us to approximate $\ln g(1)$ by a small (logarithmic) degree Taylor polynomial of $\ln g(z)$ computed at $z = 0$, provided there are no roots of $g(z)$ in the disc $D_\beta = \{z : |z| \leq \beta\}$ in the complex plane for some radius $\beta > 1$. We will need to construct a similar approximation under a weaker assumption that $g(z) \neq 0$ in a thin strip aligned with the positive real axis, that is, $g(z) \neq 0$ provided

$$-\delta \leq \Re z \leq 1 + \delta \quad \text{and} \quad |\Im z| \leq \delta$$

for some $\delta > 0$. We achieve this by constructing a polynomial $\phi = \phi_\delta : \mathbb{C} \longrightarrow \mathbb{C}$ such that

$$\begin{gathered}
\phi(0) = 0, \quad \phi(1) = 1 \quad \text{and}\\
-\delta \leq \Re\,\phi(z) \leq 1 + \delta, \quad |\Im\,\phi(z)| \leq \delta \quad \text{provided} \quad |z| \leq \beta
\end{gathered}$$

for some $\beta = \beta(\delta) > 1$, see Fig. 2.9.

We then consider a composition $h(z) = g(\phi(z))$. Hence $h(z)$ is a polynomial of $\deg h = (\deg g)(\deg \phi)$ that does not have zeros in the disc D_β and such that $h(0) = g(0)$ and $h(1) = g(1)$. Using Lemma 2.2.1, we approximate $\ln g(1)$ by the Taylor polynomial of $\ln h(z)$ of degree $n = O(\ln \deg g + \ln \deg \phi)$ computed at $z = 0$. As follows from Sect. 2.2.2, to compute the Taylor polynomial of degree n of $\ln h(z)$ at $z = 0$, it suffices to compute the Taylor polynomial of $h(z)$ of degree n at $z = 0$. On the other hand, to compute the Taylor polynomial of $h(z)$ of degree

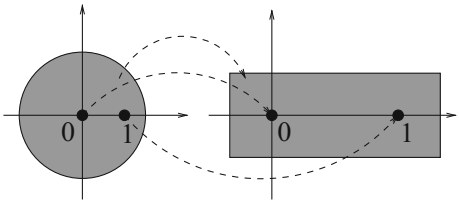

Fig. 2.9 A polynomial ϕ mapping the disc of radius $\beta > 1$ into a neighborhood of $[0, 1] \subset \mathbb{C}$ while mapping 0 to 0 and 1 to 1

n at $z = 0$ it suffices to compute the Taylor polynomial p_n of degree n of $g(z)$ at 0, compute the truncation ϕ_n of ϕ by discarding all monomials of degree higher than n and then compute the composition $p_n(\phi_n(z))$ and discard all monomials of degree higher than n (recall that $\phi(0) = 0$ so that the smallest degree of a monomial in $\phi(z)$ is 1).

The following lemma provides an explicit construction of ϕ.

2.2.3 Lemma. *For $0 < \rho < 1$, let us define*

$$\begin{aligned}
\alpha =&\alpha(\rho) = 1 - e^{-\frac{1}{\rho}}, \quad \beta = \beta(\rho) = \frac{1 - e^{-1-\frac{1}{\rho}}}{1 - e^{-\frac{1}{\rho}}} \ > \ 1, \\
N =&N(\rho) = \left\lfloor \left(1 + \frac{1}{\rho}\right) e^{1+\frac{1}{\rho}} \right\rfloor \ \geq \ 14, \quad \sigma = \sigma(\rho) = \sum_{m=1}^{N} \frac{\alpha^m}{m} \quad \textit{and} \\
\phi(z) =&\phi_\rho(z) = \frac{1}{\sigma} \sum_{m=1}^{N} \frac{(\alpha z)^m}{m}.
\end{aligned}$$

Then $\phi(z)$ is a polynomial of degree N such that $\phi(0) = 0$, $\phi(1) = 1$,

$$-\rho \ \leq \ \Re\, \phi(z) \ \leq \ 1 + 2\rho \quad \textit{and} \quad |\Im\, \phi(z)| \ \leq \ 2\rho \quad \textit{provided} \quad |z| \leq \beta.$$

Proof. Clearly, $\phi(z)$ is a polynomial of degree N such that $\phi(0) = 0$ and $\phi(1) = 1$. It remains to prove that ϕ maps the disc $|z| \leq \beta$ into the strip $-\rho \leq \Re\, z \leq 1 + 2\rho$, $|\Im\, z| \leq 2\rho$.

We consider the function

$$F_\rho(z) = \rho \ln \frac{1}{1 - z} \quad \text{for} \quad |z| < 1.$$

Since

$$\Re\, \frac{1}{1 - z} \ > \ 0 \quad \text{if} \quad |z| < 1,$$

the function $F_\rho(z)$ is well-defined by the choice of a branch of the logarithm, which we choose so that

$$F_\rho(0) = \rho \ln 1 = 0.$$

Then for $|z| < 1$ we have

$$\left|\Im\, F_\rho(z)\right| \ \leq \ \frac{\pi\rho}{2} \quad \text{and} \quad \Re\, F_\rho(z) \ \geq \ -\rho \ln 2 \tag{2.2.3.1}$$

In addition,

$$F_\rho(\alpha) = 1 \quad \text{and} \quad \Re\, F_\rho(z) \ \leq \ 1 + \rho \quad \text{provided} \quad |z| \leq 1 - e^{-1-\frac{1}{\rho}}. \tag{2.2.3.2}$$

Let

$$P_n(z) = \sum_{m=1}^{n} \frac{z^m}{m}.$$

Then

$$\left| \ln \frac{1}{1-z} - P_n(z) \right| = \left| \sum_{m=n+1}^{\infty} \frac{z^m}{m} \right| \le \frac{|z|^{n+1}}{(n+1)(1-|z|)} \quad \text{provided} \quad |z| < 1.$$

Therefore, for $|z| \le \beta$, we have

$$\begin{aligned} \left| F_\rho(\alpha z) - \rho P_N(\alpha z) \right| &\le \rho \frac{(\alpha\beta)^{N+1}}{(N+1)(1-\alpha\beta)} \\ &\le \frac{\rho}{N+1} \left(1 - e^{-1-\frac{1}{\rho}}\right)^{N+1} e^{1+\frac{1}{\rho}} \\ &\le \frac{\rho}{N+1} \le \frac{\rho}{15}. \end{aligned} \tag{2.2.3.3}$$

Combining (2.2.3.1)–(2.2.3.3), we conclude that for $|z| \le \beta$ we have

$$|\Im\, \rho P_N(\alpha z)| \le 1.64\rho \quad \text{and} \quad -0.76\rho \le \Re\, \rho P_N(\alpha z) \le 1 + 1.07\rho. \tag{2.2.3.4}$$

Substituting $z = 1$ in (2.2.3.3) and using (2.2.3.2), we conclude that

$$|1 - \rho P_N(\alpha)| \le \frac{\rho}{15}. \tag{2.2.3.5}$$

Since

$$\phi(z) = \frac{P_N(\alpha z)}{P_N(\alpha)} = \frac{\rho P_N(\alpha z)}{\rho P_N(\alpha)},$$

combining (2.2.3.4) and (2.2.3.5) and noting that $\rho P_N(\alpha)$ is real, we obtain

$$|\Im\, \phi(z)| \le 2\rho \quad \text{and} \quad -\rho \le \Re\, \phi(z) \le 1 + 2\rho \quad \text{provided} \quad |z| \le \beta.$$

□

The construction of Lemma 2.2.3 suggests a general principle:

Suppose we have a polynomial $g(z)$ of degree n such that the k-th derivative $g^{(k)}(0)$ can be computed in $n^{O(k)}$ time. We want to approximate $g(1)$. If we can find a sufficiently wide "sleeve" containing 0 and 1 and avoiding the roots of $g(z)$, such as the one on Fig. 2.10a, we can approximate $g(1)$ within relative error $0 < \epsilon < 1$ in $n^{O(\ln n - \ln \epsilon)}$ time. For that, we construct a polynomial $\phi(z)$ such that $\phi(0) = 0$, $\phi(1) = 1$ and ϕ maps the disc $\{z : |z| \le \beta\}$ for some sufficiently large $\beta > 1$ into the sleeve where $g(z) \ne 0$. We then apply Lemma 2.2.1 to $g(\phi(z))$. If the zeros of g surround 0 as on Fig. 2.10b, the sleeve connecting 0 and 1 and avoiding the roots

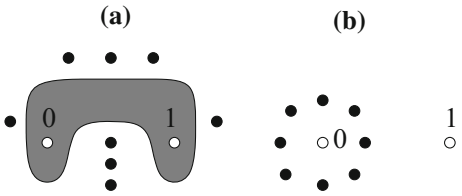

Fig. 2.10 **a** There is a sufficiently wide sleeve connecting 0 and 1 and avoiding the zeros of g, **b** the zeros of g surround 0 precluding the existence of a wide sleeve connecting 0 and 1

of $g(z)$ will have to be too thin, making the radius β of the disc too close to 1 and hence making any computational gain impossible.

2.3 Polynomials with Real Roots

We start with a definition.

2.3.1 Definition. Let f be a real polynomial of degree n with n distinct real roots $\alpha_1 < \ldots < \alpha_n$. We say that a real polynomial g of degree $n-1$ *interlaces* f if g has $n-1$ real roots $\beta_1 < \ldots < \beta_{n-1}$ such that

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \ldots < \alpha_{n-1} < \beta_{n-1} < \alpha_n,$$

see Fig. 2.11.

For example, if the roots of f are all real and distinct then f' interlaces f.

2.3.2 Theorem.

(1) Let f and $g_1, \ldots, g_m$ be real polynomials such that g_k interlaces f for $k = 1, \ldots, m$. Suppose further that the highest degree terms of $g_1, \ldots, g_m$ have the same sign. Let $\lambda_1, \ldots, \lambda_m$ be non-negative reals, not all 0 and let

$$g = \sum_{k=1}^{m} \lambda_k g_k.$$

Then the polynomial g interlaces f;

(2) Let f and g be real polynomials such that g interlaces f and suppose that the highest terms of f and g have the same sign. Then for any $\lambda \in \mathbb{R}$ the polynomial f interlaces the polynomial $h(x) = (x-\lambda)f(x) - g(x)$.

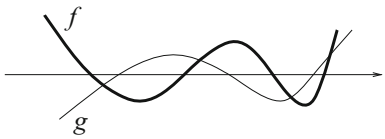

Fig. 2.11 A polynomial g interlacing a polynomial f

Proof. Let $\alpha_1 < \ldots < \alpha_n$ be the roots of f, so $\deg f = n$.

To prove Part (1), we note that since each g_k interlaces f, it changes it sign exactly once inside every interval $[\alpha_i, \alpha_{i+1}]$ for $i = 1, \ldots, n-1$, see Fig. 2.11. Since the coefficients of degree $n-1$ of all polynomials g_k have the same sign, inside each interval $[\alpha_i, \alpha_{i+1}]$ all the polynomials g_k change the sign in the same way (that is, all positive at α_i and negative at α_{i+1} or all negative at α_i and positive at α_{i+1}). It follows that g changes its sign inside each interval $[\alpha_i, \alpha_{i+1}]$ and hence interlaces f.

To Prove Part (2), without loss of generality we assume that the highest terms of f and g are positive. Since g interlaces f, the polynomial g changes its sign inside each interval $[\alpha_i, \alpha_{i+1}]$ for $i = 1, \ldots, n-1$ and hence the polynomial h also changes its sign inside each interval. Thus each interval (α_i, α_{i+1}) contains at least one root of h, which accounts for the total of $n-1$ roots.

Let $\beta_{n-1} \in (\alpha_{n-1}, \alpha_n)$ be the largest root of g. Since $g(x)$ does not change its sign for all $x > \beta_{n-1}$, we must have $g(\alpha_n) > 0$ and hence $h(\alpha_n) < 0$. On the other hand, since the highest term of $h(x)$ is positive, we must have $h(x) > 0$ for all sufficiently large x and hence there is a root, say, γ_{n+1} of $h(x)$ satisfying $\gamma_{n+1} > \alpha_n$.

Similarly, let $\beta_1 \in (\alpha_1, \alpha_2)$ be the smallest root of g. Since $g(x)$ does not change its sign for all $x < \beta_1$, we must have have $g(\alpha_1) > 0$ if n is odd (and hence $\deg g = n-1$ is even) and $g(\alpha_1) < 0$ if n is even (and hence $\deg g = n-1$ is odd). Therefore, $h(\alpha_1) < 0$ if n is odd and $h(\alpha_1) > 0$ if n is even. On the other hand, since the highest term of $h(x)$ is positive, for all sufficiently small x we must have $h(x) > 0$ if n is odd (and hence $\deg h = n+1$ is even) and $h(x) < 0$ if n is even (and hence $\deg h = (n+1)$ is odd). This proves that there is a root, say, γ_1 of $h(x)$ satisfying $\gamma_1 < \alpha_1$. Since the total number of roots of h cannot exceed $n+1$, we conclude that every interval (α_i, α_{i+1}) for $i = 1, \ldots, n-1$ contains exactly one root, say γ_{i+1} of h and hence f interlaces h. □

The coefficients of a polynomial with real roots satisfy some interesting inequalities.

2.3.3 Theorem. *Suppose that the roots of a real polynomial*

$$p(x) = \sum_{j=0}^{n} a_j x^j$$

are real. Then

$$a_j^2 \geq a_{j-1} a_{j+1} \left(1 + \frac{1}{j}\right)\left(1 + \frac{1}{n-j}\right) \quad \text{for} \quad j = 1, \ldots, n-1.$$

Equivalently, for

$$b_j = \frac{a_j}{\binom{n}{j}},$$

we have

$$b_j^2 \ \geq \ b_{j-1}b_{j+1} \quad \textit{for} \quad j = 1, \ldots, n-1.$$

Proof. Repeatedly applying Rolle's Theorem, we conclude that the roots of the polynomial

$$q(x) = \frac{d^{j-1}}{dx^{j-1}} p(x) = \sum_{k=j-1}^{n} \frac{k!}{(k-j+1)!} a_k x^{k-j+1}$$

are also real. Hence the roots of the polynomial

$$r(x) = x^{n-j+1} q\left(\frac{1}{x}\right) = \sum_{k=j-1}^{n} \frac{k!}{(k-j+1)!} a_k x^{n-k}$$

are also real. Applying Rolle's Theorem again, we conclude that the roots of the quadratic polynomial

$$\begin{aligned} s(x) =& \frac{d^{n-j-1}}{dx^{n-j-1}} r(x) \\ =& \frac{(n-j+1)!(j-1)!a_{j-1}}{2} x^2 + j!(n-j)!a_j x + \frac{(j+1)!(n-j-1)!a_{j+1}}{2} \end{aligned}$$

are real. Therefore,

$$\left(j!(n-j)!a_j\right)^2 \ \geq \ (n-j+1)!(n-j-1)!(j-1)!(j+1)!a_{j-1}a_{j+1}$$

and the proof follows. □

When the coefficients a_j are non-negative, we conclude that

$$a_j^2 \ \geq \ a_{j-1}a_{j+1} \quad \text{for} \quad j = 1, \ldots, n-1,$$

which means that the sequence $a_0, a_1, \ldots, a_n$ is *log-concave* (that is, the sequence $c_j = \ln a_j$ is concave), see [St89].

2.3.4 Estimating the largest absolute value of a root of a polynomial. Let $f(t)$ be a monic polynomial with real roots $a_1, \ldots, a_n$, so

$$f(t) = \sum_{i=0}^{n} b_i t^{n-i} = \prod_{i=1}^{n} (t - a_i),$$

where $b_0 = 1$.

Let

$$p_k = \sum_{i=1}^{n} a_i^k \quad \text{for} \quad k = 1, \ldots$$

be the power sums of roots. Knowing the $k+1$ highest coefficients $b_1, \ldots, b_{k+1}$ of f allows us to compute $p_1, \ldots, p_k$ using Newton's identities:

$$p_1 = -b_1, \quad p_2 = -b_1 p_1 - 2b_2, \quad p_3 = -b_1 p_2 - b_2 p_1 - 3b_3$$

and, more generally,

$$p_k = -k b_k - \sum_{i=1}^{k-1} b_i \, p_{k-i}.$$

On the other hand, since a_i are real, we have

$$\frac{1}{n} p_{2k} \;\leq\; \max_{i=1,\ldots,n} a_i^{2k} \;\leq\; p_{2k}.$$

In particular, by choosing $k = O(\ln(n/\epsilon))$, we can approximate $\max_{i=1,\ldots,n} |a_i|$ within a relative error ϵ by $(p_{2k})^{1/2k}$.

2.4 $\mathbb{H}$-Stable Polynomials

2.4.1 Definition. Let $f(z_1, \ldots, z_n)$ be a complex polynomial. Given a set $\Omega \subset \mathbb{C}$, we say that f is Ω-*stable* provided

$$f(z_1, \ldots, z_n) \neq 0 \quad \text{whenever} \quad z_1, \ldots, z_n \in \Omega.$$

If

$$\Omega = \{z : \Im z > 0\}$$

is the open upper half-plane, we call f $\mathbb{H}$-*stable*. In other words, f is $\mathbb{H}$-stable if

$$f(z_1, \ldots, z_n) \neq 0 \quad \text{whenever} \quad \Im z_1, \ldots, \Im z_n > 0.$$

We note that if $f(z)$ is an $\mathbb{H}$-stable univariate polynomial with real coefficients then all roots of f are necessarily real, since complex roots of f come in pairs of complex conjugate.

The following lemma summarizes some properties of $\mathbb{H}$-stable polynomials that are of critical importance for us. We follow [Wa11].

2.4.2 Lemma.

(1) Let $f_m : m = 1, \ldots,$ be a sequence of polynomials in n complex variables and let f be a polynomial such that

$$f_m \longrightarrow f$$

uniformly on compact subsets of $\mathbb{C}^n$. *If all* f_m *are* $\mathbb{H}$*-stable then either* f *is* $\mathbb{H}$*-stable or* f *is identically 0;*

(2) *Let* $f(z_1, \ldots, z_n)$ *be a* $\mathbb{H}$*-stable polynomial where* $n > 1$. *Then the polynomial*

$$g(z_1, \ldots, z_{n-1}) = f(z_1, \ldots, z_{n-1}, 0)$$

is either $\mathbb{H}$*-stable or identically 0.*

(3) *Let* $f(z_1, \ldots, z_n)$ *be a* $\mathbb{H}$*-stable polynomial and let us define*

$$g(z_1, \ldots, z_n) = \frac{\partial}{\partial z_n} f(z_1, \ldots, z_n).$$

Then either g *is* $\mathbb{H}$*-stable or* g *is identically 0.*

Proof. Part (1) follows by the (multivariate) Hurwitz Theorem which asserts that if $\Omega \subset \mathbb{C}^n$ is a connected open set, functions f_m are analytic on Ω and have no zeros in Ω, and $f_m \longrightarrow f$ uniformly on compact subsets of Ω then f either has no zeros in Ω or is identically zero in Ω (the multivariate Hurwitz Theorem immediately follows from a more standard univariate version by restricting the functions f_m and f onto a complex line in $\mathbb{C}^n$ identified with $\mathbb{C}$), see, for example, [Kr01].

To prove Part (2), we define a sequence of polynomials

$$g_m(z_1, \ldots, z_{n-1}) = f\left(z_1, \ldots, z_{n-1}, im^{-1}\right).$$

Then g_m are $\mathbb{H}$-stable for all positive integer m and $g_m \longrightarrow g$ uniformly on compact subsets of $\mathbb{C}^{n-1}$. The proof now follows by Part (1).

To prove Part (3), without loss of generality we may assume that the degree of f in z_n is $d \geq 1$, so we can write

$$f(z_1, \ldots, z_n) = \sum_{k=0}^{d} z_n^k h_k(z_1, \ldots, z_{n-1}), \tag{2.4.2.1}$$

where $h_k(z_1, \ldots, z_{n-1})$ are polynomials for $k = 0, 1, \ldots, d$ and $h_d \not\equiv 0$. Let us consider a sequence of polynomials

$$f_m(z_1, \ldots, z_n) = m^{-d} f(z_1, \ldots, z_{n-1}, mz_n) \quad \text{for} \quad m = 1, 2, \ldots.$$

Then the polynomials f_m are $\mathbb{H}$-stable and $f_m \longrightarrow z_n^d h_d(z_1, \ldots, z_{n-1})$ uniformly on compact subsets of $\mathbb{C}^n$. By Part (1), the polynomial $z_n^d h_d(z_1, \ldots, z_{n-1})$ is $\mathbb{H}$-stable and hence the polynomial $h_d(z_1, \ldots, z_{n-1})$ is $\mathbb{H}$-stable. Hence

$$h_d(z_1, \ldots, z_{n-1}) \neq 0 \quad \text{provided} \quad \Im z_1 > 0, \ldots, \Im z_{n-1} > 0. \tag{2.4.2.2}$$

Let us fix some $z_1, \dots, z_{n-1}$ such that $\Im z_1 > 0, \dots, \Im z_{n-1} > 0$ and consider a univariate polynomial

$$p(z) = f(z_1, \dots, z_{n-1}, z) \quad \text{for} \quad z \in \mathbb{C}.$$

By (2.4.2.1) and (2.4.2.2), we have $\deg p = d$. Since f is $\mathbb{H}$-stable, all the d roots (counting multiplicity) z of p satisfy $\Im z \le 0$. By the Gauss–Lucas Theorem, see Sect. 2.1.4, the roots of p' lie in the convex hull of the set of roots of p. In particular, $p'(z) \ne 0$ if $\Im z > 0$, that is,

$$g(z_1, \dots, z_{n-1}, z) \ne 0 \quad \text{provided} \quad \Im z_1 > 0, \dots, \Im z_{n-1} > 0, \Im z > 0$$

and g is $\mathbb{H}$-stable. □

Our goal is to prove the following result of Gurvits [Gu08] which bounds coefficients of an $\mathbb{H}$-stable polynomial p with non-negative real coefficients in terms of its capacity, see Sect. 2.1.5.

2.4.3 Theorem. *Let $p(x_1, \dots, x_n)$ be an $\mathbb{H}$-stable polynomial with non-negative real coefficients and such that* $\deg p \le n$. *Let us define polynomials $p_n, p_{n-1}, \dots, p_0$ by*

$$p_n = p \quad \text{and} \quad p_k = \frac{\partial}{\partial x_{k+1}} p_{k+1}\Big|_{x_{k+1}=0} \quad \text{for} \quad k = n-1, \dots, 0,$$

so that p_k is a polynomial in $x_1, \dots, x_k$ and $\deg p_k \le k$.

Suppose further that the degree of x_k in p_k does not exceed an integer d_k for $k = n, \dots, 1$.

Then

$$p_0 = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p \ge \left(\prod_{k=1}^{n} \left(\frac{d_k - 1}{d_k} \right)^{d_k - 1} \right) \inf_{x_1, \dots, x_n > 0} \frac{p(x_1, \dots, x_n)}{x_1 \cdots x_n},$$

where we agree that

$$\left(\frac{d_k - 1}{d_k} \right)^{d_k - 1} = 1 \quad \text{if} \quad d_k = 0 \quad \text{or} \quad d_k = 1.$$

The proof of Theorem 2.4.3 proceeds by induction on the number n variables with the following lemma playing the crucial role.

2.4.4 Lemma. *Let $R(t)$ be a univariate polynomial non-negative real coefficients and real roots. Suppose that* $\deg R \le d$ *for some non-negative integer d. Then*

$$R'(0) \ge \left(\frac{d-1}{d} \right)^{d-1} \inf_{t>0} \frac{R(t)}{t} \quad \text{if} \quad d > 1$$

and

$$R'(0) = \inf_{t>0} \frac{R(t)}{t} \quad \textit{if} \quad d = 1.$$

Proof. We note that

$$h(x) = \left(\frac{x-1}{x}\right)^{x-1}$$

is a decreasing function of $x > 1$. Indeed, for

$$f(x) = \ln h(x) = (x-1)\ln(x-1) - (x-1)\ln x$$

we have

$$f'(x) = \ln\frac{x-1}{x} + \frac{1}{x} < 0 \quad \text{for} \quad x > 1.$$

Therefore, without loss of generality, we may assume that $\deg R = d$.

If $d \leq 1$ then $R(t) = r_0 + r_1 t$ for some $r_0, r_1 \geq 0$ so that

$$\inf_{t>0} \frac{R(t)}{t} = r_1 = R'(0), \tag{2.4.4.1}$$

where the infimum is attained as $t \longrightarrow +\infty$.

Suppose that $d \geq 2$. If $R(0) = 0$ then $R(t) = r_1 t + \ldots + r_d t^d$ for some $r_1, \ldots, r_d \geq 0$ and we still have (2.4.4.1) where the infimum is attained as $t \longrightarrow 0+$.

Hence we may assume that $R(0) > 0$, in which case, scaling R if necessary, we may additionally assume that $R(0) = 1$. Then we can write

$$R(t) = \prod_{i=1}^{d} \left(1 - \frac{t}{\alpha_i}\right),$$

where $\alpha_1, \ldots, \alpha_d$ are the roots of R. Since the coefficients of R are non-negative and theconvex hull of the set roots $\alpha_1, \ldots, \alpha_d$ are real, we necessarily have $\alpha_1 < 0, \ldots, \alpha_d < 0$. Denoting $a_i = -\alpha_i^{-1}$, we obtain

$$R(t) = \prod_{i=1}^{d} (1 + a_i t) \quad \text{where} \quad a_1, \ldots, a_d > 0.$$

Then

$$R'(0) = a_1 + \ldots + a_d > 0$$

and applying the arithmetic-geometric mean inequality, see Sect. 2.1.1.1, we get

$$R(t) \;\le\; \left(1 + \frac{t}{d}\sum_{i=1}^{d} a_i\right)^d = \left(1 + \frac{R'(0)}{d}t\right)^d \quad \text{for} \quad t \ge 0,$$

so that

$$\inf_{t>0} \frac{R(t)}{t} \;\le\; \inf_{t>0} g(t) \quad \text{where} \quad g(t) = t^{-1}\left(1 + \frac{R'(0)}{d}t\right)^d.$$

Since $d \ge 2$ we have $g(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$ and hence the infimum of $g(t)$ is attained at a critical point t. Solving the equation $g'(t) = 0$, we get

$$t = \frac{d}{(d-1)R'(0)}$$

and

$$\inf_{t>0} \frac{R(t)}{t} \;\le\; R'(0)\left(\frac{d}{d-1}\right)^{d-1}$$

as desired. □

2.4.5 Proof of Theorem 2.4.3. From Parts (3) and (2) of Lemma 2.4.2, each polynomial p_k is either $\mathbb{H}$-stable or identically 0. We claim that

$$p_{k-1}(x_1, \ldots, x_{k-1}) \;\ge\; \left(\frac{d_k - 1}{d_k}\right)^{d_k - 1} \inf_{x_k > 0} \frac{p_k(x_1, \ldots, x_k)}{x_k}$$
$$\text{for all} \quad x_1, \ldots, x_{k-1} > 0 \tag{2.4.5.1}$$

and $k = n, n-1, \ldots, 1$ with the standard agreement that

$$\left(\frac{d_k - 1}{d_k}\right)^{d_k - 1} = 1 \quad \text{if} \quad d_k = 1 \quad \textit{or} \quad d_k = 0.$$

If p_k is identically 0 then p_{k-1} is identically 0 and (2.4.5.1) holds. Hence we assume that p_k is $\mathbb{H}$-stable.

If $k = 1$ then $p_1(x) = ax_1 + b$ for some $a, b \ge 0$ so that

$$p_0 = a = \inf_{x_1 > 0} \frac{p_1(x)}{x_1}.$$

If $k \geq 2$, for any fixed $x_1 > 0, \ldots, x_{k-1} > 0$, we define a univariate polynomial

$$R(t) = R_{x_1,\ldots,x_{k-1}}(t) = p_k(x_1, \ldots, x_{k-1}, t).$$

We claim that all the roots of R are necessarily real. Indeed, R has real coefficients and if it had a pair of complex conjugate roots $\alpha \pm \beta i$ for some $\beta > 0$ then for all sufficiently small $\epsilon > 0$ the univariate polynomial

$$\tilde{R}(t) = p_k\left(x_1 + i\epsilon, \ldots, x_{k-1} + i\epsilon, t\right)$$

would have had a root $\tilde{\alpha} + i\tilde{\beta}$ for some $\tilde{\beta} > 0$, which would have contradicted the $\mathbb{H}$-stability of p. Applying Lemma 2.4.4, we obtain

$$R'(0) \geq \left(\frac{d_k - 1}{d_k}\right)^{d_k - 1} \inf_{t>0} \frac{R(t)}{t}$$

which proves (2.4.5.1) and hence completes the proof of the theorem. □

2.4.6 Corollary. *Let p be a polynomial as in Theorem 2.4.3. Then*

$$\frac{\partial^n p}{\partial x_1 \cdots \partial x_n} \geq \frac{n!}{n^n} \inf_{x_1,\ldots,x_n>0} \frac{p\left(x_1, \ldots x_n\right)}{x_1 \cdots x_n}.$$

Proof. In Theorem 2.4.3, we can choose $d_k = k$ for $k = n, n-1, \ldots, 1$. Then

$$\prod_{k=2}^{n} \left(\frac{k-1}{k}\right)^{k-1} = \frac{(n-1)(n-2)\cdots 1}{n^{n-1}} = \frac{n!}{n^n}.$$

□

In [Gu15], Gurvits noticed that Theorem 2.4.3 leads to a bound on an arbitrary coefficient of a homogeneous $\mathbb{H}$-stable polynomial in terms of the capacity, see Sect. 2.1.5.

2.4.7 Theorem. *Let*

$$p\left(x_1, \ldots, x_n\right) = \sum_{m \in M} a_m \mathbf{x}^m$$

be an $\mathbb{H}$-stable homogeneous polynomial with non-negative coefficients. Suppose further that the degree of p in x_k does not exceed d_k for $k = 1, \ldots, n$.

Then for a non-negative integer vector $r = (\rho_1, \ldots, \rho_n)$ *such that* $\rho_1 + \ldots + \rho_n = \deg p$ *and* $\rho_k \le d_k$ *for* $k = 1, \ldots, n$, *we have*

$$a_r \ge \left(\prod_{k=1}^{n} \frac{\rho_k^{\rho_k} (d_k - \rho_k)^{d_k - \rho_k} d_k!}{\rho_k! (d_k - \rho_k)! d_k^{d_k}} \right) \operatorname{cap}_r p.$$

Proof. Without loss of generality we assume that $\rho_1, \ldots, \rho_n > 0$ since otherwise we consider the polynomial $\widehat{p}$ obtained from p by fixing $x_i = 0$ whenever $\rho_i = 0$. By Part (2) of Lemma 2.4.2, the polynomial $\widehat{p}$ is either $\mathbb{H}$-stable or identically zero (in which case the statement of the theorem trivially holds true). We define a polynomial q in d variables $y_{11}, \ldots, y_{1\rho_1}, \ldots, y_{n1}, \ldots, y_{n\rho_n}$ by

$$q\left(\ldots, y_{k1}, \ldots, y_{k\rho_k}, \ldots\right) = p\left(\ldots, \frac{y_{k1} + \ldots + y_{k\rho_k}}{\rho_k}, \ldots\right).$$

It is easy to see that q is an $\mathbb{H}$-stable of degree d and that

$$a_r = \frac{\partial^d q}{\partial y_{11} \ldots \partial y_{1\rho_1} \cdots \partial y_{n\rho_n}} \prod_{k=1}^{n} \frac{\rho_k^{\rho_k}}{\rho_k!}. \tag{2.4.7.1}$$

The degree of q in every variable y_{kj} does not exceed d_k, while the degree of the polynomial

$$\left. \frac{\partial^j q}{\partial y_{k1} \cdots \partial y_{kj}} \right|_{y_{k1} = \ldots = y_{kj} = 0}$$

in $y_{k(j+1)}$ does not exceed $d_k - j$ for $j = 1, \ldots, \rho_k$. Therefore, by Theorem 2.4.3,

$$\begin{aligned} \frac{\partial^d q}{\partial y_{11} \ldots \partial y_{1\rho_1} \cdots \partial y_{n\rho_n}} &\ge \prod_{k=1}^{n} \prod_{j=1}^{\rho_k} \left(\frac{d_k - j}{d_k - j + 1} \right)^{d_k - j} \\ &\times \inf_{\substack{y_{11}, \ldots, y_{1\rho_1} > 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ y_{n1}, \ldots, y_{n\rho_n} > 0}} \frac{q\left(y_{11}, \ldots, y_{1\rho_1}, \ldots, y_{n1}, \ldots, y_{n\rho_n}\right)}{y_{11} \cdots y_{1\rho_1} \cdots y_{n1} \cdots y_{n\rho_n}}. \end{aligned} \tag{2.4.7.2}$$

We further simplify

$$\prod_{k=1}^{n} \prod_{j=1}^{\rho_k} \left(\frac{d_k - j}{d_k - j + 1} \right)^{d_k - j} = \prod_{k=1}^{n} \frac{d_k! (d_k - \rho_k)^{d_k - \rho_k}}{(d_k - \rho_k)! d_k^{d_k}}. \tag{2.4.7.3}$$

Finally, we claim that

$$\inf_{\substack{y_{11},\ldots,y_{1\rho_1}>0\\ \ldots\ldots\ldots\ldots\\ y_{n1},\ldots,y_{n\rho_n}>0}} \frac{q\left(y_{11},\ldots,y_{1\rho_1},\ldots,y_{n1},\ldots,y_{n\rho_n}\right)}{y_{11}\cdots y_{1\rho_1}\cdots y_{n1}\cdots y_{n\rho_n}} \;\geq\; \operatorname{cap}_r p$$

$$= \inf_{x_1,\ldots,x_n>0} \frac{p\left(x_1,\ldots,x_n\right)}{x_1^{\rho_1}\cdots x_n^{\rho_n}}. \tag{2.4.7.4}$$

Indeed, given $y_{11},\ldots,y_{1\rho_1},y_{n1},\ldots,y_{n\rho_n}>0$, let us define

$$x_k=\frac{1}{\rho_k}\sum_{i=1}^{\rho_k} y_{ki} \quad \text{for} \quad k=1,\ldots,n.$$

By the arithmetic-geometric mean inequality, we have

$$x_k^{\rho_k} \;\geq\; \prod_{i=1}^{\rho_k} y_{ki}$$

and hence we obtain (2.4.7.4).

Combining (2.4.7.1)–(2.4.7.4), we get the desired result. □

2.5 $\mathbb{D}$-Stable Polynomials

Let

$$\mathbb{D}=\left\{z\in\mathbb{C}:\quad |z|\leq 1\right\}$$

be the closed unit disc. We are interested in $\mathbb{D}$-stable polynomials, that is, polynomials $p\left(z_1,\ldots,z_n\right)$ such that $p\left(z_1,\ldots,z_n\right)\neq 0$ provided $|z_i|\leq 1$ for $i=1,\ldots,n$.

We start with *multi-affine polynomials*, that is sums of square-free monomials. For a set $S\subset\{1,\ldots,n\}$, let

$$\mathbf{z}^S=\prod_{i\in S} z_i$$

denote the monomial in the complex variables $z_1,\ldots,z_n$ (we agree that $\mathbf{z}^{\emptyset}=1$). Our first result is as follows.

2.5.1 Theorem. *Let*

$$f\left(z_1,\ldots,z_n\right)=\sum_{S\subset\{1,\ldots,n\}} a_S\mathbf{z}^S \quad \text{and} \quad g\left(z_1,\ldots,z_n\right)=\sum_{S\subset\{1,\ldots,n\}} b_S\mathbf{z}^S$$

be two $\mathbb{D}$-stable polynomials. Then the polynomial $h = f * g$ defined by

$$h(z_1, \ldots, z_n) = \sum_{S \subset \{1,\ldots,n\}} c_S \mathbf{z}^S \quad \text{where} \quad c_S = a_S b_S$$

is also $\mathbb{D}$-stable.

The polynomial $h = f * g$ is called sometimes the *Schur product* and sometimes the *Hadamard product* of f and g. We follow [Hi97], see also [Ru71]. The proof is based on the *Asano contractions* [As70].

2.5.2 Lemma. *Suppose that the bivariate polynomial*

$$p(z_1, z_2) = a + bz_1 + cz_2 + dz_1 z_2$$

is $\mathbb{D}$-stable. Then the univariate polynomial

$$q(z) = a + dz$$

is also $\mathbb{D}$-stable.

Proof. Since p is $\mathbb{D}$-stable, we have $a \neq 0$. Seeking a contradiction, suppose that $q(z)$ is not $\mathbb{D}$-stable. Then $d \neq 0$ and for the unique root w of q we have $|w| = |a|/|d| \leq 1$, so that $|d| \geq |a|$.

Without loss of generality, we assume that $|b| \geq |c|$. Let us fix a z_2 such that $|z_2| = 1$ and

$$|b + dz_2| = |b| + |d| \ \geq \ |a| + |c|.$$

Then the set

$$K = \{(b + dz_2)\, z_1 : \quad |z_1| \ \leq \ 1\}$$

is a disc centered at 0 and of radius $|b| + |d| \geq |a| + |c|$. Since

$$|a + cz_2| \ \leq \ |a| + |c|,$$

the translation $K + (a + cz_2)$ of the disc K by a vector $a + cz_2$ whose length does not exceed the radius of K must contain 0, see Fig. 2.12.

Therefore, for some z_1 such that $|z_1| \leq 1$, we have $a + cz_2 + bz_1 + dz_2 z_1 = 0$, which is a contradiction. Hence $|d| < |a|$ and q is $\mathbb{D}$-stable. □

2.5.3 Proof of Theorem 2.5.1. We proceed by induction on the number n of variables. If $n = 1$, then $f(z) = a + bz$, $g(z) = c + dz$ and $h(z) = ac + bdz$. Since f is $\mathbb{D}$-stable, we have $a \neq 0$ and $|b| < |a|$. Since g is $\mathbb{D}$-stable, we have $c \neq 0$ and $|d| < |c|$. Therefore, $ac \neq 0$ and $|bd| < |ac|$, from which it follows that h is $\mathbb{D}$-stable.

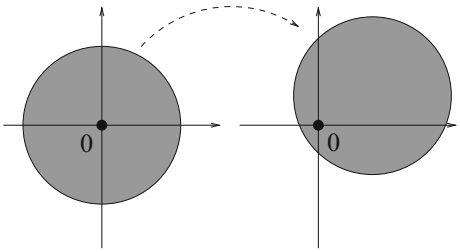

Fig. 2.12 The disc K and its translation

Suppose that $n \geq 2$. We can write

$$f(z_1, \ldots, z_n) = \sum_{S \subset \{1,\ldots,n\}} a_S \mathbf{z}^S = \sum_{S \subset \{1,\ldots,n-1\}} \left(a_S + z_n a_{S\cup\{n\}}\right) \mathbf{z}^S$$

and

$$g(z_1, \ldots, z_n) = \sum_{S \subset \{1,\ldots,n\}} b_S \mathbf{z}^S = \sum_{S \subset \{1,\ldots,n-1\}} \left(b_S + z_n b_{S\cup\{n\}}\right) \mathbf{z}^S.$$

Let us fix any two $z, w \in \mathbb{D}$. Then the $(n-1)$-variate polynomials

$$\sum_{S \subset \{1,\ldots,n-1\}} \left(a_S + z a_{S\cup\{n\}}\right) \mathbf{z}^S \quad \text{and} \quad \sum_{S \subset \{1,\ldots,n-1\}} \left(b_S + w b_{S\cup\{n\}}\right) \mathbf{z}^S$$

are $\mathbb{D}$-stable and by the induction hypothesis the polynomial

$$\sum_{S \subset \{1,\ldots,n-1\}} \left(a_S + z a_{S\cup\{n\}}\right) \left(b_S + w b_{S\cup\{n\}}\right) \mathbf{z}^S$$

in $n-1$ variables $z_1, \ldots, z_{n-1}$ is also $\mathbb{D}$-stable. This means that for any fixed $z_1, \ldots, z_{n-1} \in \mathbb{D}$ the bivariate polynomial

$$\begin{aligned} p(z, w) = &\sum_{S \subset \{1,\ldots,n-1\}} a_S b_S \mathbf{z}^S + z \sum_{S \subset \{1,\ldots,n-1\}} a_{S\cup\{n\}} b_S \mathbf{z}^S \\ &+ w \sum_{S \subset \{1,\ldots,n-1\}} a_S b_{S\cup\{n\}} \mathbf{z}^S + zw \sum_{S \subset \{1,\ldots,n-1\}} a_{S\cup\{n\}} b_{S\cup\{n\}} \mathbf{z}^S \end{aligned}$$

is $\mathbb{D}$-stable. Lemma 2.5.2 then implies that for any fixed $z_1, \ldots, z_{n-1} \in \mathbb{D}$ the univariate polynomial

$$q(z_n) = \sum_{S \subset \{1,\ldots,n-1\}} a_S b_S \mathbf{z}^S + z_n \sum_{S \subset \{1,\ldots,n-1\}} a_{S\cup\{n\}} b_{S\cup\{n\}} \mathbf{z}^S$$

is $\mathbb{D}$-stable. Therefore, for any $z_1, \ldots, z_n \in \mathbb{D}$, we have that

$$h(z_1, \ldots, z_n) = \sum_{S \subset \{1,\ldots,n-1\}} a_S b_S \mathbf{z}^S + z_n \sum_{S \subset \{1,\ldots,n-1\}} a_{S\cup\{n\}} b_{S\cup\{n\}} \mathbf{z}^S \neq 0,$$

as required. □

Ruelle [Ru71] generalized Lemma 2.5.2 as follows: let $A, B \subset \mathbb{C}$ be closed sets such that $0 \notin A$ and $0 \notin B$ and let $p(z_1, z_2) = a + bz_1 + cz_2 + dz_1z_2$ be a bivariate polynomial such that

$$p(z_1, z_2) = 0 \quad \Longrightarrow \quad z_1 \in A \quad \text{or} \quad z_2 \in B.$$

Then for the univariate polynomial $q(z) = a + dz$ we have

$$q(z) = 0 \quad \Longrightarrow \quad z = -z_1z_2 \quad \text{for some} \quad z_1 \in A \quad \text{and} \quad z_2 \in B.$$

The corresponding generalizations of Theorem 2.5.1 can be found in [Ru71] and [Hi97].

Our next goal is to prove the following theorem of Szegő for univariate $\mathbb{D}$-stable polynomials.

2.5.4 Theorem. *Let*

$$f(z) = \sum_{k=0}^{n} a_k \binom{n}{k} z^k \quad \textit{and} \quad g(z) = \sum_{k=0}^{n} b_k \binom{n}{k} z^k$$

be $\mathbb{D}$-stable polynomials. Then the polynomial $h = f * g$ defined by

$$h(z) = \sum_{k=0}^{n} c_k \binom{n}{k} z^k \quad \textit{where} \quad c_k = a_k b_k \quad \textit{for} \quad k = 0, 1, \ldots, n$$

is also $\mathbb{D}$-stable.

The polynomial $h = f * g$ is called the *Schur product* of f and g.

For $k = 0, \ldots, n$, let

$$e_k(z_1, \ldots, z_n) = \sum_{1 \le i_1 < \ldots < i_k \le n} z_{i_1} \cdots z_{i_k}$$

be the k-th elementary symmetric polynomial in $z_1, \ldots, z_n$, where we agree that $e_0(z_1, \ldots, z_n) = 1$. We deduce Theorem 2.5.4 from Theorem 2.5.1 and the following result of Szegő connecting multivariate and univariate $\mathbb{D}$-stable polynomials.

2.5.5 Theorem. *Let*

$$f(z) = \sum_{k=0}^{n} a_k \binom{n}{k} z^k$$

be a univariate $\mathbb{D}$-stable polynomial. Then the n-variate polynomial

$$F(z_1, \ldots, z_n) = \sum_{k=0}^{n} a_k e_k(z_1, \ldots, z_n)$$

is also $\mathbb{D}$-stable.

We follow Chapter IV of [Ma66] with some modifications. We start with a lemma known as *Laguerre's Theorem.*

2.5.6 Lemma. *Let $p(z)$ be a polynomial and let n be a positive integer. For $\beta \in \mathbb{C}$, we define the polynomial*

$$q(z) = np(z) + (\beta - z)p'(z).$$

(1) If $\deg p \leq n$ *then* $\deg q \leq n - 1$*;*
(2) Suppose that $\deg p \leq n$*, that p is $\mathbb{D}$-stable and that $|\beta| \leq 1$. Then q is also $\mathbb{D}$-stable.*

Proof. If $\deg p \leq n - 1$ then $\deg q \leq \deg p \leq n - 1$. If $\deg p = n$ with the highest term $a_n z^n$ then $\deg p \leq n - 1$ since the coefficient of z^n in $q(z)$ is $na_n - na_n = 0$, which completes the proof of Part (1).

By continuity, it suffices to prove Part (2) assuming that $\deg p = n$. Furthermore, without loss of generality, we assume that p is monic. Let $\alpha_1, \ldots, \alpha_n$ be the (not necessarily distinct) roots of p, each listed with its multiplicity, so that

$$p(z) = (z - \alpha_1) \cdots (z - \alpha_n) \quad \text{and} \quad |\alpha_j| > 1 \quad \text{for} \quad j = 1, \ldots, n.$$

Suppose that ζ is a root of $q(z)$. Without loss of generality, we assume that $\zeta \neq \alpha_j$ for $i = 1, \ldots, n$. Then

$$np(\zeta) + (\beta - \zeta)p'(\zeta) = 0$$

and since $p(\zeta) \neq 0$, we have $\zeta \neq \beta$ and

$$\frac{1}{\zeta - \beta} = \frac{1}{n} \frac{p'(\zeta)}{p(\zeta)} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\zeta - \alpha_i}. \tag{2.5.6.1}$$

Suppose that $|\zeta| \leq 1$. The transformation

$$z \longmapsto \frac{1}{\zeta - z} \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}$$

is a bijection between $\mathbb{C} \setminus \mathbb{D}$ and a set $S \subset \mathbb{C}$ that is either an open disc (if $|\zeta| < 1$) or an open halfplane (if $|\zeta| = 1$). In either case, S is convex. Moreover,

$$\frac{1}{\zeta - \alpha_i} \in S \quad \text{for} \quad i = 1, \ldots, n.$$

Since S is convex, by (2.5.6.1), we have

$$\frac{1}{\zeta - \beta} \in S$$

which implies that $\beta \in \mathbb{C} \setminus \mathbb{D}$, a contradiction. □

The following lemma gives a closed form description of the polynomials obtained by a repeated application of the transformation of Lemma 2.5.6.

2.5.7 Lemma. *Let*

$$f_0(z) = f(z) = \sum_{k=0}^{n} a_k \binom{n}{k} z^k$$

be a polynomial and let $\beta_1, \ldots, \beta_n$ be a sequence of complex numbers. We define polynomials $f_1, \ldots, f_n$ by

$$f_j(z) = (n - j + 1) f_{j-1}(z) + (\beta_j - z) f'_{j-1}(z) \quad \text{for} \quad j = 1, \ldots, n.$$

Then

$$f_n = n! \sum_{k=0}^{n} a_k e_k (\beta_1, \ldots, \beta_n).$$

Proof. By the repeated application of Part (1) of Lemma 2.5.6, we have $\deg f_j \leq n - j$ for $j = 0, \ldots, n$, so that f_n is a constant. We prove by induction on j that

$$f_j(z) = \sum_{k=0}^{j} \frac{(n-k)!}{(n-j)!} e_k (\beta_1 - z, \ldots, \beta_j - z) \frac{d^k}{dz^k} f(z). \tag{2.5.7.1}$$

Clearly, (2.5.7.1) holds for $j = 0$. Assuming that (2.5.7.1) holds for j, we obtain

$$\begin{aligned} f_{j+1}(z) =& (n - j) f_j(z) + (\beta_{j+1} - z) f'_j(z) \\ =& (n - j) \sum_{k=0}^{j} \frac{(n-k)!}{(n-j)!} e_k (\beta_1 - z, \ldots, \beta_j - z) \frac{d^k}{dz^k} f(z) \\ &+ (\beta_{j+1} - z) \frac{d}{dz} \sum_{k=0}^{j} \frac{(n-k)!}{(n-j)!} e_k (\beta_1 - z, \ldots, \beta_j - z) \frac{d^k}{dz^k} f(z). \end{aligned}$$

Using that for $k > 0$ we have

$$\begin{aligned}\frac{d}{dz}e_k\left(\beta_1 - z, \ldots, \beta_j - z\right) &= -\sum_{i=1}^{j} e_{k-1}\left(\ldots, \widehat{\beta_i - z}, \ldots\right)\\ &= -(j-k+1)e_{k-1}\left(\beta_1 - z, \ldots, \beta_j - z\right)\end{aligned}$$

we obtain that the coefficient of

$$\frac{d^k}{dz^k}f(z)$$

in $f_{j+1}(z)$ is

$$\begin{aligned}&\frac{(n-k)!}{(n-j-1)!}e_k\left(\beta_1 - z, \ldots, \beta_j - z\right)\\ &\quad - \frac{(n-k)!}{(n-j)!}(j-k+1)\left(\beta_{j+1} - z\right)e_{k-1}\left(\beta_1 - z, \ldots, \beta_j - z\right)\\ &\quad + \frac{(n-k+1)!}{(n-j)!}\left(\beta_{j+1} - z\right)e_{k-1}\left(\beta_1 - z, \ldots, \beta_j - z\right)\\ =&\frac{(n-k)!}{(n-j-1)!}e_k\left(\beta_1 - z, \ldots, \beta_j - z\right)\\ &\quad + \frac{(n-k)!}{(n-j-1)!}\left(\beta_{j+1} - z\right)e_{k-1}\left(\beta_1 - z, \ldots, \beta_j - z\right)\\ =&\frac{(n-k)!}{(n-j-1)!}e_k\left(\beta_1 - z, \ldots, \beta_{j+1} - z\right)\end{aligned}$$

and the proof of (2.5.7.1) follows.

Since $\deg f_n = 0$ so that $f_n(z)$ does not depend on z, from (2.5.7.1) we obtain

$$\begin{aligned}f_n = f_n(0) &= \sum_{k=0}^{n}(n-k)!e_k\left(\beta_1, \ldots, \beta_n\right)a_k\binom{n}{k}k!\\ &= n!\sum_{k=0}^{n}a_k e_k\left(\beta_1, \ldots, \beta_n\right)\end{aligned}$$

as required. □

2.5.8 Proof of Theorem 2.5.5. Let us choose arbitrary $\beta_1, \ldots, \beta_n \in \mathbb{D}$ and let us construct the polynomials f_j for $j = 1, \ldots, n$ as in Lemma 2.5.7. By the repeated application of Lemma 2.5.6, we conclude that f_n is $\mathbb{D}$-stable, that is, $f_n \neq 0$. However, by Lemma 2.5.7 we have

$$f_n = n!F\left(\beta_1, \ldots, \beta_n\right)$$

and hence $F\left(\beta_1, \ldots, \beta_n\right) \neq 0$, which completes the proof. □

2.5.9 Proof of Theorem 2.5.4. From Theorem 2.5.5 we conclude that the polynomials

$$F(z_1, \ldots, z_n) = \sum_{k=0}^{n} a_k e_k(z_1, \ldots, z_n) \quad \text{and}$$
$$G(z_1, \ldots, z_n) = \sum_{k=0}^{n} b_k e_k(z_1, \ldots, z_n)$$

are $\mathbb{D}$-stable. Since the polynomials F and G are multi-affine, by Theorem 2.5.1 the polynomial

$$H(z_1, \ldots, z_n) = \sum_{k=0}^{n} c_k e_k(z_1, \ldots, z_n)$$

is also $\mathbb{D}$-stable. Then for any $z \in \mathbb{D}$ we have

$$h(z) = H(z, \ldots, z) \neq 0$$

and hence h is $\mathbb{D}$-stable. □

We will use the following simple corollary of Theorem 2.5.4.

2.5.10 Corollary. *Let*

$$f(z) = \sum_{k=0}^{n} a_k z^k \quad \textit{and} \quad g(z) = \sum_{k=0}^{n} b_k z^k$$

be two polynomials such that $f(z) \neq 0$ *whenever* $|z| \leq \lambda$ *and* $g(z) \neq 0$ *whenever* $|z| \leq \mu$ *for some* $\lambda, \mu > 0$. *Let* $h = f * g$,

$$h(z) = \sum_{k=0}^{n} c_k z^k \quad \textit{where} \quad c_k = \frac{a_k b_k}{\binom{n}{k}} \quad \textit{for} \quad k = 0, \ldots, n.$$

Then $h(z) \neq 0$ *whenever* $|z| \leq \lambda\mu$.

Proof. The polynomials $\widehat{f}(z) = f(\lambda z)$ and $\widehat{g} = g(\mu z)$ are $\mathbb{D}$-stable. Therefore, by Theorem 2.5.4, the polynomial $\widehat{h}(z) = h(\lambda\mu z)$ is $\mathbb{D}$-stable. The proof now follows. □

For extensions and generalizations of Theorems 2.5.1 and 2.5.4, see [BB09].

Chapter 3
Permanents

Introduced in 1812 by Binet and Cauchy, permanents are of interest to combinatorics, as they enumerate perfect matchings in bipartite graphs, to physics as they compute certain integrals and to computer science as they occupy a special place in the computational complexity hierarchy. This is our first example of a partition function and we demonstrate in detail how various approaches work. Connections with $\mathbb{H}$-stable polynomials lead, in particular, to an elegant proof of the van der Waerden lower bound for the permanent of a doubly stochastic matrix. Combining it with the Bregman - Minc upper bound, we show that permanents of doubly stochastic matrices are strongly concentrated. Via matrix scaling, this leads to an efficient approximation of the permanent of non-negative matrices by a function with many convenient properties: it is easily computable, log-concave and generally amenable to analysis. As an application of the interpolation method, we show how to approximate permanents of a reasonably wide class of complex matrices and also obtain approximations of logarithms of permanents of positive matrices by low degree polynomials.

3.1 Permanents

3.1.1 Permanent. Let $A = \left(a_{ij}\right)$ be an $n \times n$ real or complex matrix. The *permanent* of A is defined as

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{3.1.1.1}$$

where S_n is the symmetric group of all $n!$ permutations of the set $\{1, \ldots, n\}$.

One can see that the permanent does not change when the rows or columns of the matrix are permuted and that per A is linear in each row and each column of A. Moreover, if $n > 1$, then denoting by A_j the $(n-1) \times (n-1)$ matrix obtained from

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Algorithms and Combinatorics 30, DOI 10.1007/978-3-319-51829-9_3

A by crossing out the first row and the j-th column, we obtain the row expansion

$$\operatorname{per} A = \sum_{j=1}^{n} a_{1j} \operatorname{per} A_j. \tag{3.1.1.2}$$

3.1.2 Permanents and perfect matchings. If A is a real matrix and $a_{ij} \in \{0, 1\}$ for all i, j then per A has a combinatorial interpretation as the number of perfect matchings in a *bipartite graph* G with *biadjacency matrix* A. Namely, the vertices of G are $1L, 2L \ldots, nL$ and $1R, 2R, \ldots, nR$ ("L" is for "left" and "R" is for "right"), whereas the edges of G are all unordered pairs $\{iL, jR\}$ for which $a_{ij} = 1$. A *perfect matching* in a graph G is a collection of edges which contain every vertex of G exactly once, see Fig. 3.1.

In this case, per A is the number of perfect matchings in G, since every perfect matching in G corresponds to a unique permutation σ such that $a_{i\sigma(i)} = 1$ for all $i = 1, \ldots, n$. For example, Fig. 3.1 pictures a graph encoded by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{3.1.2.1}$$

and a perfect matching corresponding to the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \tag{3.1.2.2}$$

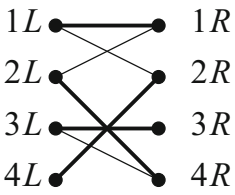


Fig. 3.1 A bipartite graph and a perfect matching (thick edges)

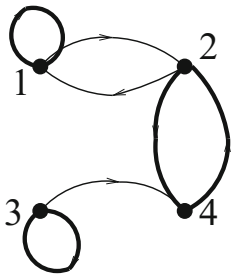


Fig. 3.2 A graph and a cycle cover (thick edges)

3.1.3 Permanents and cycle covers. A different interpretation of the permanent of a 0–1 matrix A arises if we interpret A as the *adjacency matrix* of a directed graph G. In this case, the vertices of G are $1, \ldots, n$ whereas the edges of G are all ordered pairs (i, j) such that $a_{ij} = 1$ (in particular, we allow loops). A *cycle cover* of G is a collection of edges which contain every vertex of G exactly once as the beginning point of an edge and exactly once as an endpoint of an edge, see Fig. 3.2.

In this case, per A is the number of cycle covers of G, since every cycle cover of G corresponds to a unique permutation σ such that $a_{i\sigma(i)} = 1$ for all $i = 1, \ldots, n$. For example, Fig. 3.2 pictures a graph encoded by the matrix (3.1.2.1) and a cycle cover corresponding to the permutation (3.1.2.2).

Interpretations of Sects. 3.1.2 and 3.1.3 explain why permanents are of interest to combinatorics, see [LP09] for more.

3.1.4 Permanents as integrals. Let μ_n be the Gaussian probability measure on the complex vector space $\mathbb{C}^n$ with density

$$\frac{1}{\pi^n} e^{-\|z\|^2} \quad \text{where} \quad \|z\|^2 = |z_1|^2 + \ldots + |z_n|^2 \quad \text{for} \quad z = (z_1, \ldots, z_n)\,.$$

The measure μ_n is normalized in such a way that

$$\mathbf{E}\,|z_i|^2 = 1 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad \mathbf{E}\, z_i \overline{z_j} = 0 \quad \text{for} \quad i \neq j.$$

Let $f_1, \ldots, f_n; g_1, \ldots, g_n : \mathbb{C}^n \longrightarrow \mathbb{C}$ be linear forms and let us define an $n \times n$ matrix $A = \big(a_{ij}\big)$ by

$$a_{ij} = \mathbf{E}\, f_i \overline{g_j} = \int_{\mathbb{C}^n} f_i(z) \overline{g_j(z)}\, d\mu_n \quad \text{for all} \quad i, j.$$

Then

$$\mathbf{E}\ (f_1 \cdots f_n \overline{g_1 \cdots g_n}) = \operatorname{per} A. \tag{3.1.4.1}$$

Formula (3.1.4.1) is known as (a version of) *Wick's formula*, see for example, [Zv97] and [Gu04]. To prove it, we note that both sides of (3.1.4.1) are linear in each f_i and antilinear in each g_j. Namely, denoting the left hand side of (3.1.4.1) by $L\,(f_1, \ldots, f_n; g_1, \ldots, g_n)$ and the right hand side by $R\,(f_1, \ldots, f_n; g_1, \ldots, g_n)$, we observe that

$$\begin{aligned} &L\left(f_1, \ldots, f_{i-1}, \alpha_1 f_i' + \alpha_2 f_i'', f_{i+1}, \ldots, f_n; g_1, \ldots, g_n\right) \\ &\quad = \alpha_1 L\left(f_1, \ldots, f_{i-1}, f_i', f_{i+1}, \ldots, f_n; g_1, \ldots, g_n\right) \\ &\qquad\quad + \alpha_2 L\left(f_1, \ldots, f_{i-1}, f_i'', f_{i+1}, \ldots, f_n; g_1, \ldots, g_n\right) \\ &\qquad \text{and} \\ &R\left(f_1, \ldots, f_{i-1}, \alpha_1 f_i' + \alpha_2 f_i'', f_{i+1}, \ldots, f_n; g_1, \ldots, g_n\right) \end{aligned}$$

$$= \alpha_1 R\left(f_1, \ldots, f_{i-1}, f_i', f_{i+1}, \ldots, f_n; g_1, \ldots, g_n\right) + \alpha_2 R\left(f_1, \ldots, f_{i-1}, f_i'', f_{i+1}, \ldots, f_n; g_1, \ldots, g_n\right)$$

as well as

$$\begin{aligned} &L\left(f_1, \ldots, f_n; g_1, \ldots, g_{i-1}, \alpha_1 g_i' + \alpha_2 g_i'', g_{i+1} \ldots, g_n\right) \\ &\quad = \overline{\alpha_1} L\left(f_1, \ldots, f_n; g_1, \ldots, g_{i-1}, g_i', g_{i+1}, \ldots, g_n\right) \\ &\qquad + \overline{\alpha_2} L\left(f_1, \ldots, f_n; g_1, \ldots, g_{i-1}, g_i'', g_{i+1}, \ldots, g_n\right) \\ &\qquad \text{and} \\ &R\left(f_1, \ldots, f_n; g_1, \ldots, g_{i-1}, \alpha_1 g_i' + \alpha_2 g_i'', g_{i+1}, \ldots, g_n\right) \\ &\quad = \overline{\alpha_1} R\left(f_1, \ldots, f_n; g_1, \ldots, g_{i-1}, g_i', g_{i+1}, \ldots, g_n\right) \\ &\qquad + \overline{\alpha_2} R\left(f_1, \ldots, f_n; g_1, \ldots, g_{i-1}, g_i'', g_{i+1}, \ldots, g_n\right). \end{aligned}$$

Hence it suffices to check (3.1.4.1) when each f_i and g_j is a coordinate function. Suppose therefore that

$$(f_1, \ldots, f_n) = \left(\underbrace{z_1, \ldots, z_1}_{m_1 \text{ times}}, \ldots, \underbrace{z_n, \ldots, z_n}_{m_n \text{ times}}\right) \quad \text{and}$$

$$(g_1, \ldots, g_n) = \left(\underbrace{z_1, \ldots, z_1}_{k_1 \text{ times}}, \ldots, \underbrace{z_n, \ldots, z_n}_{k_n \text{ times}}\right),$$

where $m_1, \ldots, m_n$ and $k_1, \ldots, k_n$ are non-negative integers such that

$$m_1 + \ldots + m_n = k_1 + \ldots + k_n = n.$$

If we have $m_i \neq k_i$ for some i then the left hand side of (3.1.4.1) is 0 since

$$\mathbf{E}\, z_i^{m_i} \overline{z_i^{k_i}} = 0 \quad \text{provided} \quad m_i \neq k_i.$$

On the other hand, the right hand side of (3.1.4.1) is also 0. Indeed, without loss of generality, we may assume that $m_i > k_i$. The matrix A contains an $m_i \times (n - k_i)$ block of 0s and if $m_i > k_i$ each of the $n!$ terms of (3.1.1.1) contains and least one entry from that block and hence is 0. Thus it remains to prove (3.1.4.1) in the case when $m_i = k_i$ for all $i = 1, \ldots, n$. Since

$$\mathbf{E}\, z_i^{m_i} \overline{z_i^{m_i}} = m_i!,$$

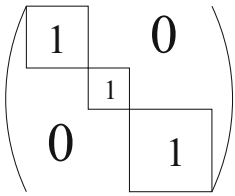


Fig. 3.3 The structure of matrix A

we conclude that the left hand side of (3.1.4.1) is $m_1! \cdots m_n!$. The matrix A in this case consists of the diagonal blocks filled by 1 s of sizes $m_1, \ldots, m_n$, see Fig. 3.3, and hence the right hand side of (3.1.4.1) is also $m_1! \cdots m_n!$. □

One immediate corollary of (3.1.4.1) is that

$$\text{per } A \ \geq \ 0 \quad \text{provided } A \text{ is Hermitian positive semidefinite.} \tag{3.1.4.2}$$

Indeed, any such $A = \big(a_{ij}\big)$ can be written as

$$a_{ij} = \mathbf{E}\left(f_i \overline{f_j}\right) \quad \text{for all} \quad i, j$$

and some linear forms $f_1, \ldots, f_n$, in which case by (3.1.4.1) we have

$$\text{per } A = \mathbf{E}\left(f_1 \cdots f_n \overline{f_1} \cdots \overline{f_n}\right) = \mathbf{E}\left(|f_1|^2 \cdots |f_n|^2\right) \geq 0.$$

The identity of Sect. 3.1.4 has some relevance to statistics of bosons in quantum physics, see, for example, [AA13] and [Ka16].

3.1.5 Permanents in computational complexity. Permanents occupy a special place in the theory of computational complexity. Valiant [Va79] proved that computing permanents of 0–1 matrices exactly (that is, counting perfect matchings in bipartite graphs exactly) is an example of a **#P**-complete problem, that is, counting perfect matchings in bipartite graphs in polynomial time exactly would lead to a polynomial time counting of the number of acceptable computations of a general non-deterministic polynomial time Turing machine, see also [AB09] and [Go08]. This is especially striking since finding whether there exists a perfect matching in a given bipartite graph is a famous problem solvable in polynomial time, see for example, [LP09]. Exact computation of permanents of 0–1 matrices leads by interpolation to exact computation of permanents of matrices with 0 and ± 1 entries and those turn out to be sufficient to encode rather involved computations. In the algebraic complexity theory, permanents stand out as universal polynomials, see Part 5 of [B+97].

Permanents also stand out as an example of the problem where randomized algorithms so far substantially outperform deterministic algorithms. The Monte Carlo Markov Chain algorithm of Jerrum, Sinclair and Vigoda [J+04] approximates permanents of non-negative matrices in polynomial time and none of the deterministic algorithms could achieve that so far, see also Sects. 3.7 and 3.9 below.

3.2 Permanents of Non-negative Matrices and $\mathbb{H}$-Stable Polynomials

3.2.1 Permanents and products of linear forms. Let $A = (a_{ij})$ be an $n \times n$ matrix and let $z_1, \ldots, z_n$ be complex variables. The following simple formula has many important consequences:

$$\operatorname{per} A = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} z_j \right). \tag{3.2.1.1}$$

In other words, per A is the coefficient of $z_1 \cdots z_n$ in the product (3.2.1.1) of linear forms.

We note that if $A = (a_{ij})$ is a non-negative real matrix with non-zero rows, then the polynomial

$$f(z_1, \ldots, z_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} z_j \right)$$

is $\mathbb{H}$-stable, see Sect. 2.4, since

$$\Im \left(\sum_{j=1}^{n} a_{ij} z_j \right) > 0 \quad \text{provided} \quad \Im z_j > 0 \quad \text{for} \quad j = 1, \ldots, n.$$

More generally, let $a_1, \ldots, a_n$ be the columns of A, so that $A = [a_1, \ldots, a_n]$. Given a non-negative integer vector $m = (m_1, \ldots, m_n)$ such that $m_1 + \ldots + m_n = n$, let

$$A_m = \left[\underbrace{a_1, \ldots, a_1}_{m_1 \text{ times}}, \ldots, \underbrace{a_k, \ldots, a_k}_{m_k \text{ times}}, \ldots, \underbrace{a_n, \ldots, a_n}_{m_n \text{ times}} \right]$$

be the $n \times n$ matrix with columns consisting of m_k copies of a_k for $k = 1, \ldots, n$. Then

$$\frac{\partial^n}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} z_j \right) = \operatorname{per} A_m \tag{3.2.1.2}$$

(if $m_k = 0$ for some k then the corresponding partial derivative is missing and so are the copies of a_k in A_m). Indeed, the left hand side of (3.2.1.2) is the coefficient of $z_1^{m_1} \cdots z_n^{m_n}$ in the product of linear forms

$$f_i(z_1, \ldots, z_n) = \sum_{j=1}^{n} a_{ij} z_j,$$

multiplied by $m_1! \cdots m_n!$. Hence the left hand side of (3.2.1.2) can be written as

$$\int_{\mathbb{C}^n} f_1 \cdots f_n \overline{z_1^{m_1} \cdots z_n^{m_n}}\, d\mu_n,$$

for the Gaussian measure μ_n of Sect. 3.1.4, and (3.2.1.2) follows by (3.1.4.1).

3.2.2 Alexandrov - Fenchel inequalities. One immediate application of (3.2.1.1) and (3.2.1.2) is an inequality for permanents of non-negative matrices, which is a particular case of the Alexandrov - Fenchel inequality for mixed volumes of convex bodies, see, for example, [Sa93].

Let $[a_1, \ldots, a_n]$ denote the $n \times n$ matrix with non-negative real columns $a_1, \ldots, a_n$. Then

$$\mathrm{per}^2[a_1, \ldots, a_n] \;\geq\; \mathrm{per}[a_1, a_1, a_3, \ldots, a_n]\,\mathrm{per}[a_2, a_2, a_3, \ldots, a_n]. \qquad (3.2.2.1)$$

By continuity, it suffices to prove (3.2.2.1) assuming that the coordinates of $a_1, \ldots, a_n$ are strictly positive. Let $a_{ij} > 0$ be the i-th coordinate of a_j. Then, from Sect. 3.2.1, the polynomial

$$f(z_1, \ldots, z_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} z_j \right)$$

is $\mathbb{H}$-stable. Let

$$g(z_1, z_2) = \frac{\partial^{n-2}}{\partial z_3 \cdots \partial z_n} f = u z_1^2 + 2 v z_1 z_2 + w z_2^2.$$

Using (3.2.1.2) we observe that

$$u = \frac{1}{2}\,\mathrm{per}[a_1, a_1, a_3, \ldots, a_n], \quad v = \frac{1}{2}\,\mathrm{per}[a_1, \ldots, a_n] \quad \text{and}$$
$$w = \frac{1}{2}\,\mathrm{per}[a_2, a_2, a_3, \ldots, a_n].$$

By the repeated application of Part (3) of Lemma 2.4.2, the quadratic polynomial q is $\mathbb{H}$-stable, which implies that $v^2 \geq uw$ and we get (3.2.2.1). Indeed, if $v^2 < uw$ then the univariate polynomial $t \longmapsto u + 2vt + wt^2$ has a pair of complex conjugate roots $\alpha \pm \beta i$ for some $\beta > 0$. Then, for any $\epsilon > 0$, the point $z_1 = 1 + i\epsilon, z_2 = (\alpha + \beta i)(1 + i\epsilon)$ is a root of $q(z_1, z_2)$ and if $\epsilon > 0$ is sufficiently small, we have $\Im z_2 = \alpha\epsilon + \beta > 0$, which contradicts the $\mathbb{H}$-stability of q.

The connection of (3.2.2.1) to the Alexandrov - Fenchel inequality for mixed volumes is as follows. Let $K_1, \ldots, K_n \subset \mathbb{R}^n$ be convex bodies and let $\lambda_1, \ldots, \lambda_n$ be positive real numbers. We consider a combination $\lambda_1 K_1 + \ldots + \lambda_n K_n$, where

$$\lambda K = \{\lambda x : x \in K\}$$

is the dilation/contraction by a factor of λ and "+" stands for the Minkowski sum of convex bodies:

$$A + B = \{x + y : x \in A, y \in B\}.$$

As is known, the volume vol $(\lambda_1 K_1 + \ldots + \lambda_n K_n)$ is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_n$ and its coefficient

$$V(K_1, \ldots, K_n) = \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n} \text{vol}\,(\lambda_1 K_1 + \ldots + \lambda_n K_n)$$

is called the *mixed volume* of $K_1, \ldots, K_n$. The Alexandrov - Fenchel inequality asserts that

$$V^2(K_1, \ldots, K_n) \geq V(K_1, K_1, K_3, \ldots, K_n) V(K_2, K_2, K_3, \ldots, K_n). \quad (3.2.2.2)$$

We obtain (3.2.2.1), if we choose K_j to be the parallelepiped, that is the direct product of axis-parallel intervals:

$$K_j = [0,\ a_{1j}] \times \ldots \times [0,\ a_{nj}].$$

In this case $\lambda_1 K_1 + \ldots + \lambda_n K_n$ is the parallelepiped

$$\left[0,\ \sum_{j=1}^{n} a_{1j}\lambda_j\right] \times \ldots \times \left[0,\ \sum_{j=1}^{n} a_{nj}\lambda_j\right],$$

cf. Fig. 3.4,
so that

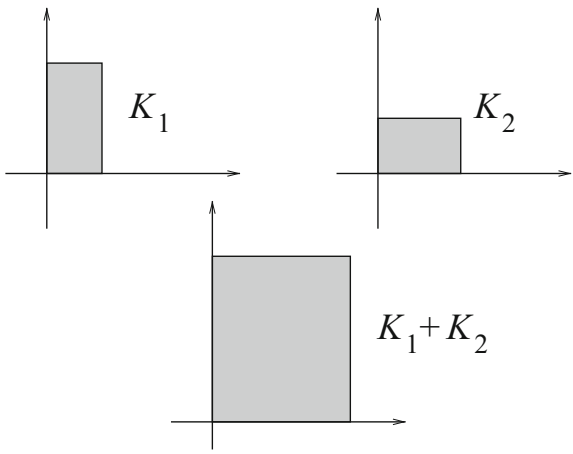


Fig. 3.4 Parallelepipeds K_1, K_2 and their Minkowski sum $K_1 + K_2$

$$\operatorname{vol}(\lambda_1 K_1 + \ldots + \lambda_n K_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \lambda_j \right)$$

and

$$V(K_1, \ldots, K_n) = \operatorname{per} A \quad \text{where} \quad A = \left(a_{ij}\right),$$

We note that for general convex bodies $K_1, \ldots, K_n$, the polynomial $\operatorname{vol}(\lambda_1 K_1 + \ldots + \lambda_n K_n)$ does not have to be $\mathbb{H}$-stable, cf. [Kh84].

3.3 The van der Waerden Inequality and Its Extensions

3.3.1 Doubly stochastic matrices. A real $n \times n$ matrix $A = \left(a_{ij}\right)$ is called *doubly stochastic* if

$$\sum_{j=1}^{n} a_{ij} = 1 \quad \text{for} \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} a_{ij} = 1 \quad \text{for} \quad j = 1, \ldots, n$$

and

$$a_{ij} \geq 0 \quad \text{for all} \quad i, j.$$

In words: a matrix is doubly stochastic if it is non-negative real with all row and column sums equal 1.

Clearly, permutation matrices (matrices, containing in each row and column exactly one non-zero entry equal to 1) are doubly stochastic, as well as the matrix

$$\frac{1}{n} J_n,$$

where J_n is the $n \times n$ matrix of all 1s.

The main goal of this section is to prove the following result, known as the van der Waerden conjecture.

3.3.2 Theorem. *Let A be an $n \times n$ doubly stochastic matrix. Then*

$$\operatorname{per} A \ \geq \ \frac{n!}{n^n}.$$

Moreover, the equality is attained if and only if $A = \dfrac{1}{n} J_n$.

Theorem 3.3.2 was first proved by Falikman [Fa81] and Egorychev [Eg81] (earlier Friedland [Fr79] proved a slightly weaker bound $\operatorname{per} A \ \geq \ e^{-n}$). Our exposition

follows Gurvits' paper [Gu08] with some simplifications introduced in [Wa11] and [LS10]. We use the notion of capacity, see Sect. 2.1.5, Theorem 2.4.3 and Corollary 2.4.6.

3.3.3 Lemma. *Let $A = \left(a_{ij}\right)$ be an $n \times n$ doubly stochastic matrix and let*

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right).$$

Then

$$\inf_{x_1, \ldots, x_n > 0} \frac{p\,(x_1, \ldots, x_n)}{x_1 \cdots x_n} = 1.$$

Proof. Clearly, $p(1, \ldots, 1) = 1$ and hence the infimum does not exceed 1. On the other hand, using the arithmetic-geometric mean inequality, see Sect. 2.1.1.1, we conclude that for $x_1, \ldots, x_n > 0$ we get

$$\prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \geq \prod_{i=1}^{n} \left(\prod_{j=1}^{n} x_j^{a_{ij}} \right) = \prod_{j=1}^{n} \left(\prod_{i=1}^{n} x_j^{a_{ij}} \right) = \prod_{j=1}^{n} \left(x_j^{\sum_{i=1}^{n} a_{ij}} \right) = \prod_{j=1}^{n} x_j$$

and hence the infimum is at least 1. □

To prove the van der Waerden inequality, we use $\mathbb{H}$-stability, see Sect. 3.2.

3.3.4 Proof of Theorem 3.3.2. As in Sect. 3.2.1, we define a polynomial $p = p_A$ in n variables $x_1, \ldots, x_n$:

$$p\,(x_1, \ldots, x_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right).$$

As we discussed in Sect. 3.2.1, the polynomial p is $\mathbb{H}$-stable and hence by Corollary 2.4.6, we have

$$\frac{\partial^n p}{\partial x_1 \cdots \partial x_n} \geq \frac{n!}{n^n} \inf_{x_1, \ldots, x_n > 0} \frac{p\,(x_1, \ldots, x_n)}{x_1 \cdots x_n}. \tag{3.3.4.1}$$

By (3.2.1.1), the left hand side of (3.3.4.1) is per A, while by Lemma 3.3.3, the infimum in the right hand side of (3.3.4.1) is 1.

In the uniqueness proof, we follow [LS10]. Suppose now that A is a doubly stochastic matrix such that $\operatorname{per} A = n!/n^n$. Then inequality (3.3.4.1) is, in fact, equation. Analyzing the proof of Theorem 2.4.3 in Sect. 2.4.5, we conclude that for

$$q(x_1, \ldots, x_{n-1}) = \frac{\partial}{\partial x_n} \left(\prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \right) \Bigg|_{x_n = 0} = \sum_{k=1}^{n} a_{kn} \prod_{i : i \neq k} \left(\sum_{j=1}^{n-1} a_{ij} x_j \right),$$

we must have

$$\inf_{x_1,\ldots,x_{n-1}>0} \frac{q\,(x_1,\ldots,x_{n-1})}{x_1\cdots x_{n-1}} = \left(\frac{n-1}{n}\right)^{n-1}. \tag{3.3.4.2}$$

Applying the arithmetic-geometric mean inequality, see Sect. 2.1.1.1, we conclude that for all $x_1 > 0, \ldots, x_{n-1} > 0$, we get

$$\begin{aligned} q(x_1,\ldots,x_{n-1}) &\geq \prod_{k=1}^{n}\prod_{i:i\neq k}\left(\sum_{j=1}^{n-1} a_{ij}x_j\right)^{a_{kn}} = \prod_{i=1}^{n}\prod_{k:k\neq i}\left(\sum_{j=1}^{n-1} a_{ij}x_j\right)^{a_{kn}} \\ &= \prod_{i=1}^{n}\left(\sum_{j=1}^{n-1} a_{ij}x_j\right)^{1-a_{in}}. \end{aligned}$$

Using the arithmetic-geometric mean inequality again, we conclude that for all $x_1 > 0, \ldots, x_{n-1} > 0$, we have

$$\begin{aligned} q(x_1,\ldots,x_{n-1}) &\geq \prod_{i=1}^{n}\left((1-a_{in})\sum_{j=1}^{n-1}\frac{a_{ij}}{1-a_{in}}x_j\right)^{1-a_{in}} \\ &\geq \prod_{i=1}^{n}\left((1-a_{in})^{1-a_{in}}\prod_{j=1}^{n-1}x_j^{a_{ij}}\right) \\ &= \left(\prod_{i=1}^{n}(1-a_{in})^{1-a_{in}}\right)\left(\prod_{j=1}^{n-1}x_j\right). \end{aligned}$$

Therefore,

$$\inf_{x_1,\ldots,x_{n-1}>0} \frac{q\,(x_1,\ldots x_{n-1})}{x_1\ldots x_{n-1}} \geq \prod_{i=1}^{n}(1-a_{in})^{1-a_{in}}.$$

By (3.3.4.2), we must have

$$\prod_{i=1}^{n}(1-a_{in})^{1-a_{in}} \leq \left(\frac{n-1}{n}\right)^{n-1}. \tag{3.3.4.3}$$

Now, since the function $t \longmapsto t\ln t$ is strictly convex for $t > 0$, see Sect. 2.1.1.2, we conclude that

$$\frac{1}{n}\sum_{i=1}^{n} t_i \ln t_i \geq \frac{t_1+\ldots+t_n}{n}\ln\frac{t_1+\ldots+t_n}{n}$$

for all $t_1, \dots, t_n$ with equality if and only if $t_1 = \dots = t_n$. Applying it with $t_i = 1 - a_{in}$, we get

$$\frac{1}{n}\sum_{i=1}^{n}(1-a_{in})\ln(1-a_{in}) \;\geq\; \frac{n-1}{n}\ln\frac{n-1}{n}$$

with equality if and only if $a_{in} = 1/n$ for $i = 1, \dots, n$. In other words,

$$\prod_{i=1}^{n}(1-a_{in})^{1-a_{in}} \;\geq\; \left(\frac{n-1}{n}\right)^{n-1}$$

with equality if and only if $a_{in} = 1/n$ for $i = 1, \dots, n$. Comparing this with (3.3.4.3), we conclude that if per $A = n!/n$, we must have $a_{in} = 1/n$ for $i = 1, \dots, n$. Since the matrix obtained from a doubly stochastic matrix by a permutation of columns remains doubly stochastic with the same permanent, we conclude that $a_{ij} = 1/n$ for all i and j as desired. □

3.3.5 Sharpening. Suppose that A is a doubly stochastic matrix and that, additionally, the j-th column of A contains not more than k_j non-zero entries for some $1 \leq k_j \leq n$ and $j = 1, \dots, n$. Using Theorem 2.4.3, we obtain

$$\operatorname{per} A = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p \;\geq\; \prod_{j=1}^{n}\left(\frac{k_j-1}{k_j}\right)^{k_j-1} \tag{3.3.5.1}$$

or, even sharply,

$$\operatorname{per} A \;\geq\; \prod_{j=1}^{n}\left(\frac{\min\{j,k_j\}-1}{\min\{j,k_j\}}\right)^{\min\{j,k_j\}-1}, \tag{3.3.5.2}$$

where the corresponding factor is 1 if $\min\{j, k_j\} = 1$. Inequalities (3.3.5.1) and (3.3.5.2) are also due to Gurvits [Gu08]. In the case when all $k_j = 3$ for all j, the inequality (3.3.5.2) was obtained by Voorhoeve [Vo79] and in the case when all k_j are equal, the inequality (3.3.5.1) was obtained by Schrijver [Sc98]. In the case of all k_j equal, we will give a different proof of (3.3.5.1) in the particular case when the non-zero entries of A are $1/k$ in Theorem 5.3.6, where we also show, following Csikvári [Cs14], that asymptotically, as n grows, the bound is logarithmically exact.

3.4 The Bregman–Minc Inequality and Its Corollaries

The following inequality was conjectured by Minc, cf. [Mi78], and proved by Bregman [Br73]. We follow the approach of Radhakrishnan [Ra97], only using the language of partitions instead that of random variables.

3.4.1 Theorem. *Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ij} \in \{0, 1\}$ for all i, j. Let*

$$r_i = \sum_{j=1}^{n} a_{ij}$$

be the number of 1 s in the ith row of A. Then

$$\operatorname{per} A \ \leq \ \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$

Let us define

$$\Omega = \{\sigma \in S_n : \ a_{i\sigma(i)} = 1 \quad \text{for} \quad i = 1, \ldots, n\}.$$

Hence

$$\operatorname{per} A = |\Omega|.$$

Without loss of generality, we assume that $\Omega \neq \emptyset$, in which case we consider Ω as a probability space with uniform measure.

We start with a probabilistic argument.

3.4.2 Lemma. *Let us fix a permutation $\sigma \in \Omega$ and an index $1 \leq i \leq n$. Let us choose a permutation $\tau \in S_n$ uniformly at random, find k such that $\tau(k) = i$ and cross out from A the columns indexed by $\sigma(\tau(1)), \ldots, \sigma(\tau(k-1))$. Let x be the number of 1 s remaining in the ith row of A after the columns are crossed out. Then*

$$\mathbf{Pr}\,(x = a) = \frac{1}{r_i} \quad \textit{for} \quad a = 1, \ldots, r_i.$$

Proof. Let J be the set of indices of columns where the ith row of A contains 1 and let $I = \sigma^{-1}(J)$. Then $i \in I$ and x is the number of indices in $\tau^{-1}(I)$ that are greater than or equal to $k = \tau^{-1}(i)$. Since $\tau \in S_n$ is chosen uniformly at random, $\tau^{-1}(i)$ is equally probable to be the largest, second largest, etc. element of $\tau^{-1}(I)$. □

3.4.3 Proof of Theorem 3.4.1

For a permutation $\tau \in S_n$ we construct a family of partitions

$$\mathcal{F}_{\tau,0} \preceq \mathcal{F}_{\tau,1} \preceq \ldots \preceq \mathcal{F}_{\tau,n}$$

of Ω as follows. We let $\mathcal{F}_{\tau,0} = \{\Omega\}$. The partition $\mathcal{F}_{\tau,1}$ consists of the events

$$F_i = \big\{\sigma \in \Omega : \ \sigma(\tau(1)) = i\big\} \quad \text{for} \quad i = 1, \ldots, n$$

(note that not more than $r_{\tau(1)}$ of the events F_i are non-empty). Generally, the partition $\mathcal{F}_{\tau,k}$ consists of the events

$$F_{i_1,\dots,i_k} = \{\sigma \in \Omega : \ \sigma(\tau(1)) = i_1, \dots, \sigma(\tau(k)) = i_k\}$$
$$\text{for distinct} \quad 1 \le i_1, \dots, i_k \le n$$

(again, some of the events can be empty). In particular, the non-empty events in $\mathcal{F}_{\tau,n}$ are singletons. From (2.1.2.4), using that $H(\{\Omega\}) = 0$ and $H(\{\mathcal{F}_{\tau,n}\}) = \ln|\Omega|$, we obtain

$$\ln|\Omega| = \sum_{k=1}^{n} H(\mathcal{F}_{\tau,k}|\mathcal{F}_{\tau,k-1}).$$

Averaging over all $\tau \in S_n$, we obtain

$$\ln|\Omega| = \frac{1}{n!}\sum_{\tau\in S_n}\sum_{k=1}^{n} H(\mathcal{F}_{\tau,k}|\mathcal{F}_{\tau,k-1}). \tag{3.4.3.1}$$

For a permutation $\sigma \in \Omega$, let $F_{\tau,k-1}(\sigma)$ be the block of $\mathcal{F}_{\tau,k-1}$ that contains σ. We consider $F_{\tau,k-1}(\sigma)$ as a probability space with conditional probability measure and let $\mathcal{F}_{\tau,k-1}(\sigma)$ be the partition of that space by the events of $\mathcal{F}_{\tau,k}$. Then

$$H(\mathcal{F}_{\tau,k}|\mathcal{F}_{\tau,k-1}) = \sum_{\sigma\in\Omega} \mathbf{Pr}(\sigma) H\big(\mathcal{F}_{\tau,k-1}(\sigma)\big),$$

cf. (2.1.2.3), and by (3.4.3.1) we have

$$\begin{aligned}\ln|\Omega| &= \frac{1}{n!}\sum_{\tau\in S_n}\sum_{k=1}^{n}\sum_{\sigma\in\Omega} \mathbf{Pr}(\sigma) H\big(\mathcal{F}_{\tau,k-1}(\sigma)\big)\\ &= \sum_{\sigma\in\Omega} \mathbf{Pr}(\sigma)\frac{1}{n!}\sum_{\tau\in S_n}\sum_{k=1}^{n} H\big(\mathcal{F}_{\tau,k-1}(\sigma)\big).\end{aligned} \tag{3.4.3.2}$$

We fix an arbitrary $\sigma \in \Omega$ and consider the sum

$$\frac{1}{n!}\sum_{\tau\in S_n}\sum_{k=1}^{n} H\big(\mathcal{F}_{\tau,k-1}(\sigma)\big). \tag{3.4.3.3}$$

Recall that $\mathcal{F}_{\tau,k-1}(\sigma)$ is the partition of the probability space Ω consisting of all permutations $\pi \in \Omega$ such that $\pi(\tau(1)) = \sigma(\tau(1)), \dots, \pi(\tau(k-1)) = \sigma(\tau(k-1))$ into the events defined by the choice of $\pi(\tau(k))$. We rearrange (3.4.3.3) in accordance with the value of $i = \tau(k)$:

$$\frac{1}{n!}\sum_{\tau\in S_n}\sum_{k=1}^{n} H\big(\mathcal{F}_{\tau,k-1}(\sigma)\big) = \sum_{i=1}^{n}\frac{1}{n!}\sum_{\tau\in S_n} H\big(\mathcal{F}_{\tau,\tau^{-1}(i)-1}(\sigma)\big) \tag{3.4.3.4}$$

and consider each term

$$\frac{1}{n!}\sum_{\tau\in S_n} H\left(\mathcal{F}_{\tau,\tau^{-1}(i)-1}(\sigma)\right) \tag{3.4.3.5}$$

separately.

Now, the partition $\mathcal{F}_{\tau,\tau^{-1}(i)-1}(\sigma)$ looks as follows. We fixed $\sigma\in\Omega$ and $1\le i\le n$. For the permutation τ, we find k such that $\tau(k)=i$, consider the probability space of all permutations $\pi\in\Omega$ such that $\pi(\tau(1))=\sigma(\tau(1)),\dots,\pi(\tau(k-1))=\sigma(\tau(k-1))$ endowed with uniform probability measure and partition it according to the value of $\pi(i)$. By (2.1.2.2),

$$H\left(\mathcal{F}_{\tau,\tau^{-1}(i)-1}(\sigma)\right) \le \ln a \quad \text{provided } \mathcal{F}_{\tau,\tau^{-1}(i)-1}(\sigma) \text{ contains } a \text{ events.}$$

By Lemma 3.4.2, the value of (3.4.3.5) does not exceed

$$\frac{1}{r_i}\sum_{a=1}^{r_i}\ln a = \frac{1}{r_i}\ln(r_i!).$$

Then by (3.4.3.4), the value of (3.4.3.3) does not exceed

$$\sum_{i=1}^{n}\frac{1}{r_i}\ln(r_i!).$$

By (3.4.3.2), we get

$$\ln|\Omega| \le \sum_{i=1}^{n}\frac{1}{r_i}\ln(r_i!),$$

and the proof follows. □

3.4.4 Remark. Let J_r be the $r\times r$ matrix filled with 1s. If A is a block-diagonal matrix with blocks $J_{r_1},\dots,J_{r_m}$, then

$$\operatorname{per} A = \prod_{i=1}^{m} r_i!,$$

from which it follows that the bound of Theorem 3.4.1 is sharp.

Theorem 3.4.1 allows us to bound permanents of stochastic matrices.

3.4.5 Corollary. *Suppose that $A=\left(a_{ij}\right)$ is an $n\times n$ stochastic matrix, that is, $a_{ij}\ge 0$ for all i, j and*

$$\sum_{j=1}^{n} a_{ij} = 1 \quad \textit{for all} \quad i = 1, \ldots, n. \tag{3.4.5.1}$$

Suppose that

$$a_{ij} \;\le\; \frac{1}{b_i} \quad \textit{for all} \quad i, j \tag{3.4.5.2}$$

and some positive integers $b_1, \ldots, b_n$. *Then*

$$\operatorname{per} A \;\le\; \prod_{i=1}^{n} \frac{(b_i!)^{1/b_i}}{b_i}.$$

Proof. Let us fix all but the i-th row of an $n \times n$ matrix A and allow the ith row vary. Then per A is a linear function in the i-th row $a_i = (a_{i1}, \ldots, a_{in})$. Let us consider the polytope P_i of all n-vectors $a_i = (a_{i1}, \ldots, a_{in})$ such that all entries a_{ij} are non-negative and the conditions (3.4.5.1) and (3.4.5.2) are met. By linearity, the maximum value of per A on P_i is attained at a vertex of P_i, in which case we necessarily have $a_{ij} \in \{0, 1/b_{ij}\}$ for $j = 1, \ldots, n$. Indeed, if $0 < a_{ij_1} < 1/b_i$ for some j_1 then there is another $j_2 \neq j_1$ such that $0 < a_{ij_2} < 1/b_i$ (recall that b_i is an integer). In that case, we can write $a_i = \left(a_i^1 + a_i^2\right)/2$, where a_i^1 is obtained from a_i by the perturbation $a_{ij_1} := a_{ij_1} + \epsilon$, $a_{ij_2} := a_{ij_2} - \epsilon$ and a_i^2 is obtained from a_i by the perturbation $a_{ij_1} := a_{ij_1} - \epsilon$, $a_{ij_2} := a_{ij_2} + \epsilon$ for a sufficiently small $\epsilon > 0$, which implies that a_i is not a vertex of P_i.

Hence we conclude that the maximum of per A on the set of $n \times n$ non-negative matrices $A = \left(a_{ij}\right)$ satisfying (3.4.5.1) and (3.4.5.2) is attained when $a_{ij} \in \{0, 1/b_{ij}\}$ for all i, j. Let B be the matrix obtained from such a matrix A by multiplying the i-th row by b_i. Then

$$\operatorname{per} B = \left(\prod_{i=1}^{n} \frac{1}{b_i}\right) \operatorname{per} A \quad \text{and} \quad \operatorname{per} B \;\le\; \prod_{i=1}^{n} (b_i!)^{1/b_i}$$

by Theorem 3.4.1. □

The author learned Corollary 3.4.5 and its proof from A. Samorodnitsky [Sa01], see also [So03] for a somewhat more general statement with b_i not required to be integer.

3.4.6 Concentration of the permanent of doubly stochastic matrices. The van der Waerden bound (Theorem 3.3.2) together with the Bregman - Minc bound (Corollary 3.4.5) implies that per A does not vary much if A is a doubly stochastic matrix with small entries. Indeed, suppose that A is an $n \times n$ doubly stochastic matrix. Then, by Theorem 3.3.2,

$$\operatorname{per} A \;\ge\; \frac{n!}{n^n} \;\ge\; e^{-n}.$$

Let us fix an $\alpha \ge 1$ and suppose that, additionally,

$$a_{ij} \leq \frac{\alpha}{n} \quad \text{for all} \quad i, j.$$

Let

$$b = \left\lfloor \frac{n}{\alpha} \right\rfloor,$$

so that

$$a_{ij} \leq \frac{1}{b} \quad \text{for all} \quad i, j$$

and by Corollary 3.4.5,

$$\text{per } A \leq \left(\frac{(b!)^{1/b}}{b} \right)^n = e^{-n} n^{O(\alpha)}.$$

Hence if the entries of an $n \times n$ doubly stochastic matrix are within a constant factor of each other, the permanent of the matrix varies within a polynomial in n factor.

In fact,

$$\prod_{i,j=1}^{n} \left(1 - a_{ij}\right)^{1-a_{ij}} \leq \text{per } A \leq 2^n \prod_{i,j=1}^{n} \left(1 - a_{ij}\right)^{1-a_{ij}} \tag{3.4.6.1}$$

for any $n \times n$ doubly stochastic matrix A (if $a_{ij} = 1$ the corresponding factor is 1), where the lower bound is due to Schrijver [Sc98] and the upper bound was recently established by Gurvits and Samorodnitsky [GS14], who also conjectured that the upper bound holds with 2^n replaced by $2^{n/2}$.

The following useful inequality was conjectured by Vontobel [Vo13] and deduced by Gurvits [Gu11] from the lower bound in (3.4.6.1)

Let $A = \left(a_{ij}\right)$ be an $n \times n$ positive real matrix and let $B = \left(b_{ij}\right)$ be an $n \times n$ doubly stochastic matrix.Then

$$\ln \text{per } A \geq \sum_{i,j=1}^{n} b_{ij} \ln \frac{a_{ij}}{b_{ij}} + \sum_{i,j=1}^{n} \left(1 - b_{ij}\right) \ln \left(1 - b_{ij}\right).$$

We prove the inequality in Theorem 5.4.2 following the approach of Lelarge [Le15]. Note that if A is doubly stochastic, by choosing $B = A$ we recover the lower bound in (3.4.6.1).

3.5 Matrix Scaling

Results of Sects. 3.3 and 3.4 provide us with some rather useful estimates of permanents of doubly stochastic matrices. It turns out that computing the permanent of any *positive* real matrix can be easily reduced to computing the permanent of a doubly stochastic matrix.

3.5.1 Matrix scaling. Let $A = (a_{ij})$ be an $n \times n$ matrix. We say that A is obtained by *scaling* from an $n \times n$ matrix $B = (b_{ij})$ if

$$a_{ij} = \lambda_i \mu_j b_{ij} \quad \text{for all} \quad i, j$$

and some numbers $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$.

We note that in this case

$$\operatorname{per} A = \left(\prod_{i=1}^{n} \lambda_i \right) \left(\prod_{j=1}^{n} \mu_j \right) \operatorname{per} B. \tag{3.5.1.1}$$

3.5.2 Theorem. *For any $n \times n$ matrix $A = (a_{ij})$ such that*

$$a_{ij} > 0 \quad \textit{for all} \quad i, j,$$

there exists a unique $n \times n$ doubly stochastic matrix $B = (b_{ij})$ and positive $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ such that

$$a_{ij} = \lambda_i \mu_j b_{ij} \quad \textit{for all} \quad i, j. \tag{3.5.2.1}$$

The numbers λ_i and μ_j are unique up to a rescaling

$$\lambda_i \longmapsto \lambda_i \tau, \quad \mu_j \longmapsto \mu_j \tau^{-1}$$

for some $\tau > 0$.

Proof. Without loss of generality, we may assume that $n \geq 2$. Let Ω_n be the polytope of all $n \times n$ doubly stochastic matrices $X = (x_{ij})$ and let us consider a function $f : \Omega_n \longrightarrow \mathbb{R}$ defined by

$$f(X) = \sum_{i,j=1}^{n} x_{ij} \ln \frac{x_{ij}}{a_{ij}}.$$

Then f is a strictly convex function, cf. Sect. 2.1.1.2, and hence it attains its unique minimum, say $B = (b_{ij})$, on Ω_n.

First, we establish that $b_{ij} > 0$ for all i, j. Indeed,

$$\frac{\partial}{\partial x_{ij}} f(X) = \ln \frac{x_{ij}}{a_{ij}} + 1. \tag{3.5.2.2}$$

If $x_{ij} = 0$ we consider the right derivative and conclude that it is equal to $-\infty$, while for any $x_{ij} > 0$ the derivative is finite. Let $\frac{1}{n} J_n \in \Omega_n$ be the matrix with all entries equal to $1/n$ and let $B(t) = (1-t)B + t\frac{1}{n}J_n$, so that $B(0) = B$ and $B(1) = \frac{1}{n}J_n$. If $b_{ij} = 0$ for some i, j then for all sufficiently small $t > 0$ we have

$$f(B_t) < f(B),$$

which contradicts the definition of B as the minimum point of f.

Thus B is a positive matrix and therefore lies in the relative interior of Ω_n. It follows from (3.5.2.2) by the Lagrange multiplier conditions that there are numbers $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ such that

$$\ln \frac{b_{ij}}{a_{ij}} = \alpha_i + \beta_j \quad \text{for all} \quad i, j.$$

Letting

$$\lambda_i = e^{-\alpha_i} \quad \text{and} \quad \mu_j = e^{-\beta_j},$$

we obtain (3.5.2.1).

On the other hand, if a doubly stochastic matrix $B = \left(b_{ij}\right)$ satisfies (3.5.2.1) then necessarily $b_{ij} > 0$ for all i, j and B is a critical point of f on Ω_n. Since f is strictly convex, B must be the unique minimum point of f on Ω_n, which proves the uniqueness of B.

From (3.5.2.1) and the uniqueness of B, we obtain the uniqueness of λ_i and μ_j up to a rescaling. □

Scaling can be obtained by solving a different optimization problem.

3.5.3 Lemma. *Let $A = \left(a_{ij}\right)$ be an $n \times n$ positive matrix. Let us define a function $g_A : \mathbb{R}^n \oplus \mathbb{R}^n \longrightarrow \mathbb{R}$ by*

$$g_A(x, y) = \sum_{i,j=1}^{n} a_{ij} e^{x_i + y_j} \quad \textit{where} \quad x = (x_1, \ldots, x_n) \quad \textit{and} \quad y = (y_1, \ldots, y_n)$$

and let $\mathcal{L} \subset \mathbb{R}^n \oplus \mathbb{R}^n$ be the subspace defined by the equations

$$\sum_{i=1}^{n} x_i = \sum_{j=1}^{n} y_j = 0.$$

Then g attains its minimum on $\mathcal{L}$ at some point (x^*, y^*) where $x^* = (\xi_1, \ldots, \xi_n)$ and $y^* = (\eta_1, \ldots, \eta_n)$. Let

$$\lambda_i = e^{-\xi_i}\sqrt{\frac{g_A(x^*, y^*)}{n}} \quad \text{and} \quad \mu_j = e^{-\eta_j}\sqrt{\frac{g_A(x^*, y^*)}{n}}$$

for all i, j and let us define an $n \times n$ matrix $B = \left(b_{ij}\right)$ by

$$b_{ij} = \lambda_i^{-1}\mu_j^{-1}a_{ij} \quad \text{for all} \quad i, j.$$

Then B is a doubly stochastic matrix.

Proof. First, we claim that the minimum of g_A on $\mathcal{L}$ is indeed attained at some point. Let

$$\delta = \min_{ij} a_{ij} > 0.$$

Since for all $(x, y) \in \mathcal{L}$, we have $x_i \geq 0$ and $y_j \geq 0$ for some i and j, we have

$$g_A(x, y) > g_A(0, 0) \quad \text{if} \quad x_i > \ln\frac{g_A(0, 0)}{\delta} \quad \text{or} \quad y_j > \ln\frac{g_A(0, 0)}{\delta}$$

for some i, j. On the other hand, if for some $(x, y) \in \mathcal{L}$ we have $x_i < -t$ for some $t > 0$ then $x_j > t/n$ for some j and, similarly, if $y_i < -t$ for some $t > 0$ then $y_j > t/n$ for some j. Therefore, the minimum of g_A on $\mathcal{L}$ is attained on the compact subset

$$|x_i|, \left|y_j\right| \leq n \ln\frac{g_A(0, 0)}{\delta} \quad \text{for all} \quad i, j.$$

At the minimum point, the gradient of $g_A(x, y)$ is orthogonal to $\mathcal{L}$, so for some α and β we have

$$\begin{aligned} &\sum_{j=1}^{n} a_{ij}e^{\xi_i+\eta_j} = \alpha \quad \text{for} \quad i = 1, \ldots, n \\ &\qquad \text{and} \\ &\sum_{i=1}^{n} a_{ij}e^{\xi_i+\eta_j} = \beta \quad \text{for} \quad j = 1, \ldots, n. \end{aligned} \tag{3.5.3.1}$$

Summing the first set of equations over $i = 1, \ldots, n$ and the second set of equations over $j = 1, \ldots, n$, we conclude that

$$\sum_{i,j=1}^{n} a_{ij}e^{\xi_i+\eta_j} = n\alpha = n\beta,$$

so

$$\alpha = \beta = \frac{1}{n} g_A\left(x^*, y^*\right)$$

and the proof follows from (3.5.3.1). □

3.5.4 Remark. Theorem 3.5.2 was proved by Sinkhorn [Si64], who used a different approach. He showed that, given a positive matrix A, the repeated row and column scaling (first, scale all rows to row sum 1, then scale all columns to column sum 1, then again rows, then again columns, etc.) converges to the desired doubly stochastic matrix B. An approach to scaling via a solution of an appropriate optimization problem (similar to our Lemma 3.5.3) was used in [MO68] and several other papers since then.

Clearly, not every non-negative matrix can be scaled to doubly stochastic (for example, the matrix of all zeros cannot). Some non-negative matrices can be scaled arbitrarily close to doubly stochastic, but cannot be scaled exactly, for example the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Indeed, multiplying the first column by $\epsilon > 0$ and the first row by ϵ^{-1}, we obtain the matrix

$$B = \begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix}$$

with row and column sums arbitrarily close to 1, but never exactly 1. It is shown in [L+00] that a non-negative matrix A can be scaled arbitrarily close to a doubly stochastic matrix if and only if $\operatorname{per} A > 0$ and that it can be scaled exactly to a doubly stochastic matrix, if, in addition, whenever for a set $I \subset \{1, \ldots, n\}$ of rows and for a set $J \subset \{1, \ldots, n\}$ of columns such that $|I| + |J| = n$ we have $a_{ij} = 0$ for $i \in I$ and $j \in J$, we must also have $a_{ij} = 0$ for all $i \notin I$ and $j \notin J$. The conditions for approximate and exact scaling can be efficiently (in polynomial time) verified. Also [L+00] contains the fastest known algorithm for matrix scaling.

As is observed in [L+00], formula (3.5.1.1) together with the inequality

$$\frac{n!}{n^n} \leq \operatorname{per} B \leq 1$$

for the permanent of a doubly stochastic matrix B allows one to estimate the permanent of any $n \times n$ non-negative matrix A within a multiplicative factor of roughly e^n and the inequality (3.4.6.1) improves the factor further to 2^n (and, conjecturally, to $2^{n/2}$). Computationally, matrix scaling is very efficient and in view of Sect. 3.4.6 it is natural to ask for which matrices A their doubly stochastic scaling B will not have large entries, so that a better upper bound on $\operatorname{per} B$ can be used.

3.5.5 Definition. Let $A = (a_{ij})$ be an $n \times n$ positive matrix. For $\alpha \geq 1$ we say that A is α-*conditioned* if

$$a_{ij_1} \leq \alpha a_{ij_2} \quad \text{for any} \quad 1 \leq i, j_1, j_2 \leq n$$

and

$$a_{i_1 j} \leq \alpha a_{i_2 j} \quad \text{for any} \quad 1 \leq i_1, i_2, j \leq n.$$

In words: an $n \times n$ positive matrix is α-conditioned if the ratio of any two entries of A in the same row and the ratio of any two entries of A in the same column do not exceed α.

3.5.6 Lemma. *Let A be an $n \times n$ matrix which is α-conditioned for some $\alpha \geq 1$. Let $B = (b_{ij})$ be the doubly stochastic matrix obtained from A by scaling. Then B is α^2-conditioned. In particular,*

$$b_{ij} \leq \frac{\alpha^2}{n} \quad \textit{for all} \quad i, j.$$

Proof. Let $A = (a_{ij})$ and let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ be positive real such that

$$b_{ij} = \lambda_i \mu_j a_{ij} \quad \text{for all} \quad i, j.$$

Then

$$\frac{b_{ij_1}}{b_{ij_2}} = \frac{\mu_{j_1}}{\mu_{j_2}} \frac{a_{ij_1}}{a_{ij_2}} \leq \frac{\mu_{j_1}}{\mu_{j_2}} \alpha \quad \text{for all} \quad 1 \leq j_1, j_2 \leq n. \tag{3.5.6.1}$$

Since

$$\sum_{i=1}^{n} b_{ij_1} = \sum_{i=1}^{n} b_{ij_2} = 1,$$

we conclude that

$$\frac{\mu_{j_1}}{\mu_{j_2}} \geq \frac{1}{\alpha} \quad \text{for all} \quad j_1, j_2.$$

On the other hand, since

$$\frac{a_{ij_1}}{a_{ij_2}} \geq \frac{1}{\alpha} \quad \text{for all} \quad j_1, j_2,$$

from (3.5.6.1) we conclude that

$$\frac{b_{ij_1}}{b_{ij_2}} \geq \frac{1}{\alpha^2} \quad \text{for all} \quad j_1, j_2. \tag{3.5.6.2}$$

Similarly, we prove that

$$\frac{b_{i_1 j}}{b_{i_1 j}} \geq \frac{1}{\alpha^2} \quad \text{for all} \quad i_1, i_2$$

and hence B is α^2-conditioned.

Since

$$\sum_{j=1}^{n} b_{ij} = 1 \quad \text{for all} \quad i = 1, \ldots, n,$$

we have

$$b_{ij} \geq \frac{1}{n} \quad \text{for every} \quad i \quad \text{and some} \quad j$$

and the proof follows by (3.5.6.2). □

Lemma 3.5.6 together the observation of Sect. 3.4.6 and formula (3.5.1.1) allows us, given an $n \times n$ positive matrix A whose entries are within a constant factor of each other, to compute per A by scaling within a polynomial in n factor.

Although the scaling factors $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ are not uniquely defined by the matrix, Theorem 3.5.2 implies that their product $\lambda_1 \cdots \lambda_n \mu_1 \cdots \mu_n$ is a function of the matrix. It has some interesting convex properties.

3.5.7 Lemma. *For an $n \times n$ positive matrix $A = (a_{ij})$, let us define a number $f(A)$ as follows: Let $B = (b_{ij})$ be a doubly stochastic matrix and let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ be positive numbers such that*

$$a_{ij} = \lambda_i \mu_j b_{ij} \quad \textit{for all} \quad i, j.$$

Let

$$f(A) = \left(\prod_{i=1}^{n} \lambda_i \right) \left(\prod_{j=1}^{n} \mu_j \right).$$

Then f is well-defined and satisfies the following properties:

(1) Function f is homogeneous of degree n:

$$f(\alpha A) = \alpha^n f(A) \quad \textit{for all} \quad \alpha > 0$$

and all positive $n \times n$ matrices A;

(2) Function f is monotone:

$$f(C) \leq f(A)$$

for any positive $n \times n$ matrices $A = (a_{ij})$ and $C = (c_{ij})$ such that

$$c_{ij} \leq a_{ij} \quad \textit{for all} \quad i, j;$$

(3) Function $f^{1/n}$ is concave:

$$f^{1/n}(\alpha_1 A_1 + \alpha_2 A_2) \geq \alpha_1 f^{1/n}(A_1) + \alpha_2 f^{1/n}(A_2)$$

for any positive $n \times n$ matrices A_1 and A_2 and any $\alpha_1, \alpha_2 \geq 0$ such that $\alpha_1 + \alpha_2 = 1$.

Proof. Theorem 3.5.2 implies that f is well-defined and Part (1) is straightforward. As in Lemma 3.5.3, let us define

$$g_A(x, y) = \sum_{i,j=1}^{n} a_{ij} e^{x_i + y_j}$$

and let $\mathcal{L} \subset \mathbb{R}^n \oplus \mathbb{R}^n$ be the subspace defined by the equations $x_1 + \ldots + x_n = 0$ and $y_1 + \ldots + y_n = 0$. Then, by Lemma 3.5.3,

$$f(A) = \frac{1}{n^n} \min_{(x,y)\in\mathcal{L}} g_A^n(x, y).$$

Since $g_C(x, y) \leq g_A(x, y)$ for all $(x, y) \in \mathcal{L}$ provided $c_{ij} \leq a_{ij}$ for all i, j, the proof of Part (2) follows.

We have

$$f^{1/n}(A) = \frac{1}{n} \min_{(x,y)\in\mathcal{L}} g_A(x, y)$$

and hence for $A = \alpha_1 A_1 + \alpha_2 A_2$ we have

$$\begin{aligned} f^{1/n}(A) =& \frac{1}{n} \min_{(x,y)\in\mathcal{L}} g_A(x, y) = \frac{1}{n} \min_{(x,y)\in\mathcal{L}} \alpha_1 g_{A_1}(x, y) + \alpha_2 g_2 A_2(x, y) \\ \geq& \frac{\alpha_1}{n} \min_{(x,y)\in\mathcal{L}} g_{A_1}(x, y) + \frac{\alpha_2}{n} \min_{(x,y)\in\mathcal{L}} g_{A_2}(x, y) = \alpha_1 f^{1/n}(A_1) + \alpha_2 f^{1/n}(A_2), \end{aligned}$$

which completes the proof of Part (3). □

It is not hard to see that the function f of Lemma 3.5.7 is the capacity

$$\inf_{x_1,\ldots,x_n>0} \frac{p(x_1, \ldots, x_n)}{x_1 \cdots x_n}$$

of the polynomial

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right),$$

cf. Sect. 2.1.5 and Lemma 3.3.3.

We state the scaling theorem in the most general form (we will use it later in Chap. 8).

3.5.8 Theorem. *Let* $r = (r_1, \dots, r_m)$ *and* $c = (c_1, \dots, c_n)$ *be positive integer vectors such that*

$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j = N.$$

Then for any positive $m \times n$ *matrix* $A = \big(a_{ij}\big)$ *there exists an* $m \times n$ *positive matrix* $B = \big(b_{ij}\big)$ *with row sums* $r_1, \dots, r_m$ *and column sums* $c_1, \dots, c_n$ *and positive real* $\lambda_1, \dots, \lambda_m$ *and* $\mu_1, \dots, \mu_n$ *such that*

$$a_{ij} = \lambda_i \mu_j b_{ij} \quad \textit{for all} \quad i, j.$$

Moreover, given r, c *and* A, *the matrix* B *is unique and can be found as the minimum point of the function*

$$f = \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} x_{ij} \ln \frac{x_{ij}}{a_{ij}}$$

on the polytope $\Omega_{r,c}$ *of non-negative* $m \times n$ *matrices with row sums* r *and column sums* c. *The numbers* λ_i *and* μ_j *are unique up to a rescaling*

$$\lambda_i \longmapsto \lambda_i \tau, \quad \mu_j \longmapsto \mu_j \tau^{-1}$$

for some $\tau > 0$ *and can be found as follows:*

Let us define $g_A : \mathbb{R}^m \oplus \mathbb{R}^n \longrightarrow \mathbb{R}$ *by*

$$g_A(x, y) = \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} a_{ij} e^{x_i + y_j} \quad \textit{for} \quad x = (x_1, \dots, x_m) \quad \textit{and} \quad y = (y_1, \dots, y_n)$$

and let $\mathcal{L}_{r,c} \subset \mathbb{R}^m \oplus \mathbb{R}^n$ *be the subspace defined by the equations*

$$\sum_{i=1}^{m} r_i x_i = 0 \quad \textit{and} \quad \sum_{j=1}^{n} c_j y_j = 0.$$

Then the minimum of g_A *on* $\mathcal{L}_{r,c}$ *is attained at some point* $x^* = (\xi_1, \dots, \xi_m)$ *and* $y^* = (\eta_1, \dots, \eta_n)$ *and we may let*

$$\lambda_i = e^{-\xi_i} \sqrt{\frac{g_A(x^*, y^*)}{N}} \quad \textit{and} \quad \mu_j = e^{-\eta_j} \sqrt{\frac{g_A(x^*, y^*)}{N}}$$

for all i, j. □

The proof is very similar to those of Theorem 3.5.2 and Lemma 3.5.3 and therefore omitted.

3.6 Permanents of Complex Matrices

In this section, we take a look at the permanents of matrices with complex entries. Such permanents are of interest in physics, see, for example, [AA13] and [Ka16]. First, we prove that the permanents of matrices sufficiently close to the $n \times n$ matrix J_n of all 1s is not 0.

3.6.1 Theorem. *There exists an absolute constant $\delta_0 > 0$ (one can choose $\delta_0 = 0.5$) such that for any $n \times n$ matrix $A = (a_{ij})$ with complex entries satisfying*

$$|1 - a_{ij}| \leq \delta_0 \quad \text{for all} \quad i, j$$

we have

$$\operatorname{per} A \neq 0.$$

Geometrically, the ℓ^∞ distance from the matrix J_n to the hypersurface per $Z = 0$ in the space $\mathbb{C}^{n \times n}$ of $n \times n$ complex matrices is bounded below by a positive constant, independent on n. Later, in Theorem 5.5.3, we prove that per $A \neq 0$ if the ℓ^1 distance of every row and column of an $n \times n$ complex matrix A to the vector of all 1 s does not exceed γn for some absolute constant $\gamma > 0$ (one can choose $\gamma = 0.0696$).

In view of Theorem 3.6.1, we can choose a branch of $\ln \operatorname{per} A$ for all matrices $A = (a_{ij})$ satisfying $|1 - a_{ij}| \leq \delta_0$ such that $\ln \operatorname{per} J_n$ is a real number, where J_n is the $n \times n$ matrix of all 1s.

3.6.2 Theorem. *Let us fix some $0 < \delta < \delta_0$, where δ_0 is the constant in Theorem 3.6.1. Then there exists $\gamma = \gamma(\delta) > 0$ and for any $\epsilon > 0$ and positive integer n there exists a polynomial $p = p_{n,\delta,\epsilon}$ in the entries of an $n \times n$ complex matrix $A = (a_{ij})$ satisfying*

$$\deg p \leq \gamma(\ln n - \ln \epsilon)$$

and

$$|\ln \operatorname{per} A - p(A)| \leq \epsilon$$

provided

$$|1 - a_{ij}| \leq \delta \quad \text{for all} \quad i, j.$$

As we will see, the polynomial $p(A)$ can be efficiently computed. The gist of Theorem 3.6.2 is that $\ln \operatorname{per} A$ can be efficiently approximated by a low-degree polynomial in the vicinity of the matrix J_n of all 1s, and, in particular, per A can be approximated there within a relative error of ϵ in quasi-polynomial $n^{O(\ln n - \ln \epsilon)}$ time.

Theorems 3.6.1 and 3.6.2 were first proved in [B16b] with a worse constant $\delta_0 = 0.195$. Following [B16+], we give a much simplified proof achieving a better constant.

First we prove Theorem 3.6.1 and then deduce Theorem 3.6.2 from it. We identify $\mathbb{C} = \mathbb{R}^2$ and measure angles between complex numbers as vectors in the plane.

3.6.3 Lemma. *Let $u_1, \ldots, u_n \in \mathbb{R}^2$ be non-zero vectors and suppose that the angle between any two vectors u_i and u_j does not exceed α for some $0 \le \alpha < 2\pi/3$. Let $u = u_1 + \ldots + u_n$. Then*

$$|u| \ \ge \ \left(\cos\frac{\alpha}{2}\right)\sum_{i=1}^{n}|u_i|.$$

Proof. First, we note that 0 cannot lie in the convex hull of the vectors $u_1, \ldots, u_n$, since otherwise by the Carathéodory Theorem it would have lied in the convex hull of some three vectors u_i, u_j, u_k and then the angle between some two of these three vectors would have been at least $2\pi/3$, see Fig. 3.5.

Hence the vectors $u_1, \ldots, u_n$ lie in an angle measuring at most α. Let us consider the orthogonal projections of $u_1, \ldots, u_n$ onto the bisector of the angle, see Fig. 3.6.

Then the length of the projection of u_i is at least $|u_i|\cos(\alpha/2)$ and the length of the projection of u is at least $(|u_1| + \ldots + |u_n|)\cos(\alpha/2)$. Since the length of u is at least as large as the length of its orthogonal projection, the result follows. □

In [B16b] a weaker bound with $\sqrt{\cos\alpha}$ instead of $\cos(\alpha/2)$ is used (assuming that $\alpha < \pi/2$). The current enhancement is due to Bukh [Bu15].

3.6.4 Lemma. *Let $u_1, \ldots, u_n \in \mathbb{C}$ be non-zero complex numbers, such that the angle between any two vectors u_i and u_j does not exceed α for some $0 \le \alpha < 2\pi/3$ and let $0 \le \delta < \cos(\alpha/2)$ be a real number. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be complex numbers such that*

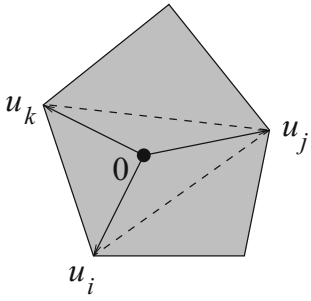


Fig. 3.5 If the origin lies in the convex hull of the vectors then the angle between some two vectors is at least $2\pi/3$

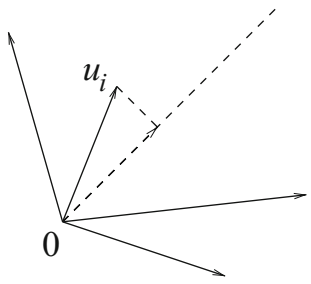


Fig. 3.6 Projecting vectors onto the bisector of the angle

$$\left|1 - a_j\right| \le \delta \quad \textit{and} \quad \left|1 - b_j\right| \le \delta \quad \textit{for} \quad j = 1, \ldots, n.$$

Let

$$v = \sum_{j=1}^{n} a_j u_j \quad \textit{and} \quad w = \sum_{j=1}^{n} b_j u_j.$$

Then $v \neq 0$, $w \neq 0$ *and the angle between* v *and* w *does not exceed*

$$2 \arcsin \frac{\delta}{\cos(\alpha/2)}.$$

Proof. Let $u = u_1 + \ldots + u_n$. Then, by Lemma 3.6.3, $u \neq 0$ and

$$|u| \ge \cos\left(\frac{\alpha}{2}\right) \sum_{j=1}^{n} |u_j|.$$

By the triangle inequality, we have

$$|v - u| \le \sum_{j=1}^{n} \left|1 - a_j\right| |u_j| \le \delta \sum_{j=1}^{n} |u_j|.$$

Therefore, the angle between $v = (v - u) + u$ and u does not exceed

$$\theta = \arcsin \frac{|v - u|}{|u|} \le \arcsin \frac{\delta}{\cos(\alpha/2)},$$

see Fig. 3.7.

Similarly, the angle between w and u does not exceed θ and hence the angle between v and w does not exceed 2θ. □

3.6.5 Proof of Theorem 3.6.1. Let us choose

$$\delta_0 = 0.5 \quad \text{and} \quad \alpha = \frac{\pi}{2}.$$

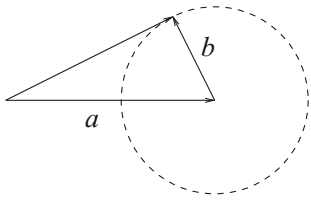


Fig. 3.7 The angle between a and $a + b$ does not exceed $\arcsin \frac{|b|}{|a|}$ provided $|b| < |a|$

We denote by $\mathcal{U}_n$ the closed polydisc $\mathcal{U}_n \subset \mathbb{C}^{n\times n}$ consisting of the $n \times n$ complex matrices $A = (a_{ij})$ such that

$$|1 - a_{ij}| \leq \delta_0 \quad \text{for all} \quad i, j.$$

We prove by induction on n the following statement.

For every matrix $Z \in \mathcal{U}_n$ we have per $Z \neq 0$ and, moreover, if $A, B \in \mathcal{U}_n$ are two matrices that differ in one row (one column) only, then the angle between non-zero complex numbers per A and per B does not exceed α.

If $n = 1$ then any $a \in \mathcal{U}_1$ is necessarily non-zero, since $\delta_0 < 1$. Moreover, the angle between any two $a, b \in \mathcal{U}_1$ does not exceed $2 \arcsin \delta_0 = \pi/3 < \alpha$, cf. Fig. 3.7.

Suppose that $n \geq 2$ and assume that the above statement holds for matrices from $\mathcal{U}_{n-1}$. Let $A, B \in \mathcal{U}_n$ be two matrices that differ in one row or in one column only. Without loss of generality, we assume that the matrix B is obtained from A by replacing the entries a_{1j} in the first row by some complex numbers b_{1j}, where $j = 1, \ldots, n$. Using the row expansion (3.1.1.2), we obtain

$$\operatorname{per} A = \sum_{j=1}^{n} a_{1j} \operatorname{per} A_j \quad \text{and} \quad \operatorname{per} B = \sum_{j=1}^{n} b_{1j} \operatorname{per} A_j,$$

where A_j is the $(n-1) \times (n-1)$ matrix obtained from A by crossing out the first row and the j-th column. We have $A_j \in \mathcal{U}_{n-1}$ and, moreover, up to a permutation of columns, any two matrices A_{j_1} and A_{j_2} differ in at most one column. Therefore, by the induction hypothesis per $A_j \neq 0$ for $j = 1, \ldots, n$ and the angle between any two non-zero complex numbers per A_{j_1} and per A_{j_2} does not exceed α.

We apply Lemma 3.6.4 with $u_j = \operatorname{per} A_j$, $a_j = a_{1j}$ and $b_j = b_{1j}$ for $j = 1, \ldots, n$. Since $\delta_0 < \cos(\alpha/2)$, by Lemma 3.6.4 we have per $A \neq 0$ and per $B \neq 0$ and the angle between per A and per B does not exceed

$$2 \arcsin \frac{\delta_0}{\cos(\alpha/2)} = 2 \arcsin \frac{0.5}{\cos(\pi/4)} = 2 \arcsin \frac{1}{\sqrt{2}} = \frac{\pi}{2} = \alpha,$$

which completes the proof. □

The value of $\delta_0 = 0.5$ is the largest value of δ for which the equation

$$\alpha = 2 \arcsin \frac{\delta}{\cos(\alpha/2)}$$

has a solution α. Indeed, the above equation can be written as

$$\left(\sin \frac{\alpha}{2}\right)\left(\cos \frac{\alpha}{2}\right) = \delta, \quad \text{that is,} \quad \sin \alpha = 2\delta.$$

3.6.6 The optimal value of δ_0. What is the optimal value of δ_0 in Theorem 3.6.1? To be more precise, since it is not even clear whether the optimal value δ_0 exists, what is the supremum of all possible values of δ_0 in Theorem 3.6.1? Since

$$\operatorname{per}\begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix} = 0$$

we must have

$$\delta_0 < \frac{\sqrt{2}}{2} \approx 0.7071067810.$$

Moreover, Bukh [Bu15] showed that for

$$a = \frac{1+i}{2} \quad \text{and} \quad b = \frac{1-i}{2}$$

we have

$$\operatorname{per} \underbrace{\begin{pmatrix} a & b & a & b & \dots & a & b \\ b & a & b & a & \dots & b & a \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a & b & a & b & \dots & a & b \\ b & a & b & a & \dots & b & a \end{pmatrix}}_{n \equiv 2 \mod 4} = 0$$

and hence there is no hope that the value of δ_0 might improve as n grows.

Now we deduce Theorem 3.6.2 from 3.6.1.

3.6.7 Proof of Theorem 3.6.2. Let $A = (a_{ij})$ be an $n \times n$ complex matrix satisfying $|a_{ij} - 1| \le \delta$ for all i, j and let $J = J_n$ be the $n \times n$ matrix of all 1s. We define a univariate polynomial

$$g(z) = \operatorname{per}\big(J + z(A - J_n)\big)$$

with $\deg g \le n$. Let

$$\beta = \frac{\delta_0}{\delta} > 1.$$

By Theorem 3.6.1,

$$g(z) \neq 0 \quad \text{provided} \quad |z| \le \beta.$$

Let

$$f(z) = \ln g(z) \quad \text{for} \quad |z| \le 1,$$

where we choose the branch of the logarithm that is real for $z = 0$. We note that by Theorem 3.6.1 the function f is well defined and we have

$$f(0) = \ln n! \quad \text{and} \quad f(1) = \ln \operatorname{per} A.$$

We consider the Taylor polynomial of f at $z = 0$:

$$p_m(z) = f(0) + \sum_{k=1}^{m} \frac{z^k}{k!} \frac{d^k}{dz^k} f(z)\Big|_{z=0.} \tag{3.6.7.1}$$

By Lemma 2.2.1, we have

$$|p_m(1) - \ln \operatorname{per} A| = |p_m(A) - f(1)| \;\le\; \frac{n}{(m+1)\beta^m(\beta-1)}$$

In particular, to approximate $\ln \operatorname{per} A$ within an additive error of $\epsilon > 0$, we can choose $m \le \gamma(\ln n - \ln \epsilon)$ in (3.6.7.1) for some $\gamma = \gamma(\delta) > 0$.

It remains to show that $p_m(1)$ is a polynomial of degree m in the matrix entries a_{ij} of A. Our first observation is that the k-th derivative $g^{(k)}(0)$ is a polynomial of degree k in the entries of the matrix A, which can be computed in $n^{O(k)}$ time. Indeed,

$$\begin{aligned}\frac{d^k}{dz^k} g(z)\Big|_{z=0} &= \frac{d^k}{dz^k} \sum_{\sigma\in S_n} \prod_{i=1}^{n} \left(1 + z\left(a_{i\sigma(i)} - 1\right)\right)\Big|_{z=0} \\ &= \sum_{\sigma\in S_n} \sum_{(i_1,\dots,i_k)} \left(a_{i_1\sigma(i_1)} - 1\right)\cdots\left(a_{i_k\sigma(i_k)} - 1\right),\end{aligned}$$

where the last sum is taken over all ordered k-subsets $(i_1, \dots, i_k)$ of indices $1 \le i_j \le n$. Since there are $(n-k)!$ permutations $\sigma \in S_n$ that map a given ordered k-subset $(i_1, \dots, i_k)$ into a given ordered k-subset $(j_1, \dots, j_k)$, we can write

$$g^{(k)}(0) = (n-k)! \sum_{\substack{(i_1,\dots,i_k)\\(j_1,\dots,j_k)}} \left(a_{i_1 j_1} - 1\right)\cdots\left(a_{i_k j_k} - 1\right), \tag{3.6.7.2}$$

where the last sum is taken over all pairs of ordered k-subsets $(i_1, \dots, i_k)$ and $(j_1, \dots, j_k)$ of indices between 1 and n. As follows from Sect. 2.2.2, the derivatives $f^{(k)}(0)$ for $k = 1, \dots, m$ can be found in $O(m^2)$ time as linear combinations of the derivatives $g^{(k)}(0)$ for $k = 1, \dots, m$ with coefficients depending on k only, which completes the proof. □

Kontorovich and Wu [KW16] implemented the algorithm of Sect. 3.6.7 for computing the polynomial $p(A)$ and performed numerical experiments. Computing $g^{(k)}(0)$ reduces to computing the sum of permanents of $k \times k$ submatrices of $A - J_n$ and Kontorovich and Wu used for that purpose an efficient algorithm of [FG06]. It turned out that for $n \times n$ matrices $A = \left(a_{ij}\right)$ satisfying $|1 - a_{ij}| \le 0.5$ and $n \le 20$ (so that the exact value of per A can be computed for comparison), polynomials p of degree 3 already provide reasonable approximations (they approximate $\ln \operatorname{per} A$ within an about 1% error). On the other hand, polynomials p of degree 3 can be easily computed for 100×100 matrices.

Let A be an $n \times n$ complex matrix such that $\operatorname{per} A \neq 0$ and suppose that the ℓ^∞-distance from A to the complex hypersurface $\operatorname{per} Z = 0$ is at least δ_0 for some $\delta_0 > 0$. It follows from the proof of Sect. 3.6.7 that for any $0 < \delta < \delta_0$ there is a constant $\gamma = \gamma(\delta) > 0$ and for any $0 < \epsilon < 1$ there is a polynomial $p = p_{A,\delta,\epsilon}$ in the entries of an $n \times n$ matrix $B = \left(b_{ij}\right)$ such that $\deg p \leq \gamma(\ln n - \ln \epsilon)$ and

$$\left|\ln \operatorname{per} B - p_{A,\delta,\epsilon}(B)\right| \leq \epsilon \quad \text{provided} \quad \left|a_{ij} - b_{ij}\right| \leq \delta \quad \text{for all} \quad i, j.$$

Of course, depending on A, the polynomial p might be hard to compute (it is easy when $A = J_n$, the matrix of all 1s).

3.6.8 Remark. If the entries of an $n \times n$ real matrix $A = \left(a_{ij}\right)$ are (weakly) decreasing down each column, that is, if $a_{ij} \geq a_{(i+1)j}$ for all i, j then the roots of the polynomial $p(z) = \operatorname{per}(J_n + zA)$ are real. Moreover, the n-variate polynomial

$$p(z_1, \ldots, z_n) = \operatorname{per}(J_n D(z_1, \ldots, z_n) + A),$$

where $D(z_1, \ldots, z_n)$ is the diagonal matrix having $z_1, \ldots, z_n$ on the diagonal, is $\mathbb{H}$-stable [B+11].

A different approach to approximation of permanents by Taylor polynomial expansions around J_n is described in [Mc14].

3.7 Approximating Permanents of Positive Matrices

As follows from Sect. 3.5, for any $\alpha \geq 1$, fixed in advance, the permanent of an α-conditioned $n \times n$ positive matrix A can be approximated in polynomial time within an $n^{O(\alpha^2)}$ factor. Understanding permanents of complex matrices allows us to approximate permanents of such matrices better: we show that we can approximate the permanent within arbitrarily small relative error in quasi-polynomial time. More precisely, we prove the following result.

3.7.1 Theorem. *For any $0 \leq \delta < 1$, there exists $\gamma = \gamma(\delta) > 0$ such that for any positive integer n and any real $0 < \epsilon \leq 1$ there exists a polynomial $p = p_{n,\delta,\epsilon}$ with $\deg p \leq \gamma\left(\ln n - \ln \epsilon\right)$ in the entries a_{ij} of an $n \times n$ real matrix $A = \left(a_{ij}\right)$ such that*

$$|\ln \operatorname{per} A - p(A)| \leq \epsilon$$

provided

$$\left|1 - a_{ij}\right| \leq \delta \quad \text{for all} \quad i, j.$$

We show that the polynomial $p_{n,\delta,\epsilon}$ can be computed in $n^{O(\ln n - \ln \epsilon)}$ time, where the implicit constant in the "O" notation depends on δ alone.

We deduce Theorem 3.7.1 from the following result.

3.7.2 Theorem. *Let us fix a real* $0 \le \delta < 1$ *and le*

$$\tau = (1-\delta)\sin\left(\frac{\pi}{4} - \arctan\delta\right) > 0.$$

Let $Z = \left(z_{ij}\right)$ *be an* $n \times n$ *complex matrix such that*

$$\left|1 - \Re\, z_{ij}\right| \le \delta \quad \textit{and} \quad \left|\Im\, z_{ij}\right| \le \tau \quad \textit{for all} \quad 1 \le i, j \le n.$$

Then

$$\operatorname{per} Z \neq 0.$$

We note that

$$(1-\delta)\sin\left(\frac{\pi}{4} - \arctan\delta\right) \ge \frac{(1-\delta)^2}{2} \quad \text{for all} \quad 0 \le \delta \le 1$$

and so

$$\tau = \frac{(1-\delta)^2}{2}$$

satisfies the condition of Theorem 3.7.2.

We prove Theorem 3.7.2 first and then deduce Theorem 3.7.1 from it.

As in Sect. 3.6, we identify $\mathbb{C} = \mathbb{R}^2$ and measure angles between non-zero complex numbers as between non-zero vectors in the plane. We start with a simple geometric lemma.

3.7.3 Lemma. *Let* $u_1, \ldots, u_n \in \mathbb{C}$ *be non-zero complex numbers such that the angle between any two* u_i, u_j *does not exceed* $\pi/2$.

(1) Let

$$v = \sum_{j=1}^{n} \alpha_j u_j \quad \textit{and} \quad w = \sum_{j=1}^{n} \beta_j u_j$$

where $\alpha_1, \ldots, \alpha_n$ *are non-negative real and* $\beta_1, \ldots, \beta_n$ *are real such that*

$$|\beta_j| \le \alpha_j \quad \textit{for} \quad j = 1, \ldots, n.$$

Then

$$|w| \le |v|;$$

(2) Let

$$v = \sum_{j=1}^{n} \alpha_j u_j \quad \textit{and} \quad w = \sum_{j=1}^{n} \beta_j u_j$$

where $\alpha_1, \ldots, \alpha_n$ *and* $\beta_1, \ldots, \beta_n$ *are real such that*

$$|1-\alpha_j| \le \delta \quad \textit{and} \quad |1-\beta_j| \le \delta \quad \textit{for} \quad j=1,\dots,n$$

and some $0 \le \delta < 1$. *Then* $v \ne 0$, $w \ne 0$ *and the angle between* v *and* w *does not exceed*

$$2\arctan\delta.$$

(3) Let

$$v=\sum_{j=1}^{n}\alpha_j u_j \quad \textit{and} \quad w=\sum_{j=1}^{n}\beta_j u_j$$

where

$$|1-\Re\,\alpha_j| \le \delta, \quad |1-\Re\,\beta_j| \le \delta \quad \textit{and}$$
$$|\Im\,\alpha_j| \le \tau, \quad |\Im\,\beta_j| \le \tau \quad \textit{for} \quad j=1,\dots,n$$

and some $0 \le \delta < 1$ *and* $0 \le \tau < 1-\delta$. *Then* $v \ne 0$, $w \ne 0$ *and the angle between* v *and* w *does not exceed*

$$2\arctan\delta + 2\arcsin\frac{\tau}{1-\delta}.$$

Proof. We consider the standard inner product in $\mathbb{R}^2=\mathbb{C}$, so

$$\langle a,b\rangle = \Re\, a\overline{b}.$$

Hence

$$\langle u_i,u_j\rangle \ge 0 \quad \text{for all} \quad i,j.$$

We have

$$|w|^2=\sum_{1\le i,j\le n}\beta_i\beta_j\langle u_i,u_j\rangle \le \sum_{1\le i,j\le n}\alpha_i\alpha_j\langle u_i,u_j\rangle = |v|^2$$

and the proof of Part (1) follows.

To prove Part (2), let

$$u=\sum_{j=1}^{n}\left(\frac{\alpha_j+\beta_j}{2}\right)u_j \quad \text{and} \quad x=\sum_{j=1}^{n}\left(\frac{\alpha_j-\beta_j}{2}\right)u_j,$$

so that $v=u+x$ and $w=u-x$, see Fig. 3.8. Clearly, $|u|>0$.

Now, if $|1-\alpha| \le \delta$ and $|1-\beta| \le \delta$ for some $0 \le \delta < 1$ and $\alpha \ge \beta$ we have

$$\frac{\alpha}{\beta} \le \frac{1+\delta}{1-\delta} \quad \text{and hence} \quad \alpha(1-\delta) \le \beta(1+\delta)$$

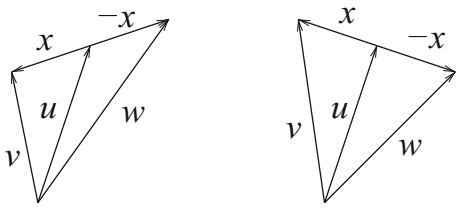


Fig. 3.8 Given $|u|$ and $|x|$, the angle between $v = u + x$ and $w = u - x$ is the largest when u is orthogonal to x

and

$$\frac{\alpha - \beta}{\alpha + \beta} - \delta = \frac{\alpha - \beta - \delta(\alpha + \beta)}{\alpha + \beta} = \frac{\alpha(1 - \delta) - \beta(1 + \delta)}{\alpha + \beta} \leq 0.$$

Therefore for all α and β such that $|1 - \alpha| \leq \delta$ and $|1 - \beta| \leq \delta$ for some $0 \leq \delta < 1$ we have

$$\frac{|\alpha - \beta|}{\alpha + \beta} \leq \delta.$$

Therefore, by Part (1),

$$|x| \leq \delta |u|.$$

The angle between v and w is

$$\arccos \frac{\langle v, w \rangle}{|v||w|},$$

where

$$\langle v, w \rangle = |u|^2 - |x|^2.$$

We have

$$|v|^2 + |w|^2 = 2|u|^2 + 2|x|^2$$

and hence

$$|v||w| \leq |u|^2 + |x|^2$$

with equality attained when $|v|^2 = |w|^2 = |u|^2 + |x|^2$, that is, when x is orthogonal to u. Therefore, the angle between v and w does not exceed

$$\arccos \frac{|u|^2 - |x|^2}{|u|^2 + |x|^2}$$

with equality attained when x is orthogonal to u and the angle is

$$2 \arctan \frac{|x|}{|u|} \leq 2 \arctan \delta,$$

see Fig. 3.8. The proof of Part (2) now follows.

In Part (3), let

$$v' = \sum_{j=1}^{n} \left(\Re\, \alpha_j\right) u_j, \quad v'' = \sum_{j=1}^{n} \left(\Im\, \alpha_j\right) u_j, \quad w' = \sum_{j=1}^{n} \left(\Re\, \beta_j\right) u_j$$
$$\text{and} \quad w'' = \sum_{j=1}^{n} \left(\Im\, \beta_j\right) u_j.$$

By Part (2), the angle between non-zero vectors v' and w' does not exceed $2 \arctan \delta$. By Part (1), we have

$$|v''| \le \frac{\tau}{1-\delta}|v'| \quad \text{and} \quad |w''| \le \frac{\tau}{1-\delta}|w'|.$$

Hence $v = v' + i\,v'' \neq 0$ and $w = w' + i\,w'' \neq 0$ and the angle between v and v' and the angle between w and w' do not exceed

$$\arcsin \frac{\tau}{1-\delta},$$

see Fig. 3.7. The proof of Part (3) now follows. □

Now we are ready to prove Theorem 3.7.2.

3.7.4 Proof of Theorem 3.7.2. For a positive integer n, let $\mathcal{U}_n = \mathcal{U}_n(\delta, \tau)$ be the set of $n \times n$ complex matrices $Z = \left(z_{ij}\right)$ such that

$$\left|1 - \Re\, z_{ij}\right| \le \delta \quad \text{and} \quad \left|\Im\, z_{ij}\right| \le \tau \quad \text{for all} \quad i, j.$$

We prove by induction on n a stronger statement:

For any $Z \in \mathcal{U}_n$ we have $\operatorname{per} Z \neq 0$ and, moreover, if $A, B \in \mathcal{U}_n$ are two matrices that differ in one row (or in one column) only, then the angle between the non-zero complex numbers $\operatorname{per} A$ and $\operatorname{per} B$ does not exceed $\pi/2$.

Since $\tau < 1 - \delta$, the statement holds for $n = 1$. Assuming that the statement holds for matrices in $\mathcal{U}_{n-1}$, let us consider two matrices $A, B \in \mathcal{U}_n$ that differ in one row or in one column only. Without loss of generality, we assume that B is obtained from A by replacing the entries a_{1j} in the first row with complex numbers b_{1j} for $j = 1, \ldots, n$. Let A_j be the $(n-1) \times (n-1)$ matrix obtained from A by crossing out the first row and the j-th column. Applying the row expansion (3.1.1.2), we get

$$\operatorname{per} A = \sum_{j=1}^{n} a_{1j} \operatorname{per} A_j \quad \text{and} \quad \operatorname{per} B = \sum_{j=1}^{n} b_{1j} \operatorname{per} A_j.$$

We have $A_j \in \mathcal{U}_{n-1}$ for all $j = 1, \ldots, n$, and, moreover any two matrices A_{j_1} and A_{j_2} differ, up to a permutation of columns, in one column only. Therefore, by the

induction hypothesis, we have per $A_j \neq 0$ for $j = 1, \ldots, n$ and the angle between any two non-zero complex numbers A_{j_1} and A_{j_2} does not exceed $\pi/2$. Applying Part (3) of Lemma 3.7.3 with

$$u_j = \operatorname{per} A_j, \quad \alpha_j = a_{1j} \quad \text{and} \quad \beta_j = b_{1j} \quad \text{for} \quad j = 1, \ldots, n,$$

we conclude that per $A \neq 0$, per $B \neq 0$ and the angle between per A and per B does not exceed

$$2 \arctan \delta + 2 \arcsin \frac{\tau}{1-\delta} = \frac{\pi}{2}.$$

□

3.7.5 Proof of Theorem 3.7.1. Let $A = \big(a_{ij}\big)$ be an $n \times n$ real matrix such that

$$\big|1 - a_{ij}\big| \ \leq \ \delta \quad \text{for all} \quad i, j,$$

let $J_n = J$ be the $n \times n$ matrix filled with 1 s and let us define a univariate polynomial

$$r(z) = \operatorname{per}\big(J + z(A - J)\big) \quad \text{for} \quad z \in \mathbb{C}.$$

Hence

$$r(0) = \operatorname{per} J = n!, \quad r(1) = \operatorname{per} A \quad \text{and} \quad \deg r \leq n.$$

First, we observe that as long as $-\alpha \ \leq \ \Re z \ \leq \ 1 + \alpha$ for some $\alpha > 0$, the real part of each entry of the matrix $J + z(A - J)$ lies in the interval

$$[1 - \delta(1 + \alpha), \ 1 + \delta(1 + \alpha)].$$

Similarly, as long as $|\Im z| \leq \rho$ for some $\rho > 0$, the imaginary part of each entry of the matrix $J + z(A - J)$ does not exceed $\rho\delta$ in the absolute value. Let us choose an $\alpha = \alpha(\delta) > 0$ such that $\delta' = \delta(1 + \alpha) < 1$ and choose

$$\rho = \rho(\delta) = \frac{1 - \delta'}{\delta} \sin\left(\frac{\pi}{4} - \arctan \delta'\right) > 0.$$

It follows from Theorem 3.7.2 that

$$r(z) \neq 0 \quad \text{provided} \quad -\alpha \ \leq \ \Re z \ \leq \ 1 + \alpha \quad \text{and} \quad |\Im z| \ \leq \ \rho. \tag{3.7.5.1}$$

Let $\phi(z) = \phi_\delta(z)$ be the univariate polynomial constructed in Lemma 2.2.3, such that

$$\phi(0) = 0, \quad \phi(1) = 1$$

and

$$-\alpha \ \leq \ \Re \phi(z) \ \leq \ 1 + \alpha \quad \text{and} \quad |\Im \phi(z)| \ \leq \ \rho$$

provided

$$|z| \leq \beta \quad \text{for some} \quad \beta = \beta(\delta) > 1.$$

The degree of $\phi(z)$ is bounded by a constant depending on δ alone.

Let us define

$$g(z) = r(\phi(z)).$$

Then $g(z)$ is a univariate polynomial and $\deg g = (\deg r)(\deg \phi) = O(n)$ where the implicit constant in the "O" notation depends only on δ. We have

$$g(0) = r(0) = n!, \quad g(1) = r(1) = \operatorname{per} A$$

and from (3.7.5.1) it follows that

$$g(z) \neq 0 \quad \text{provided} \quad |z| \leq \beta.$$

Let us choose a branch of $f(z) = \ln g(z)$ in the disc $|z| \leq 1$ so that

$$f(0) = \ln n! \quad \text{and} \quad f(1) = \ln \operatorname{per} A$$

and let p_m be the Taylor polynomial of degree m of $f(z)$ computed at $z = 0$, so

$$p_m(z) = f(0) + \sum_{k=1}^{m} \left(\frac{d^k}{dz^k} f(z) \Big|_{z=0} \right) \frac{z^k}{k!}.$$

By Lemma 2.2.1, we have

$$|f(1) - p_m(1)| \leq \frac{\deg g}{(m+1)\beta^m(\beta - 1)}.$$

Hence one can choose $m \leq \gamma\big(\ln n - \ln \epsilon\big)$ for some constant $\gamma = \gamma(\delta) > 0$ such that

$$|\ln \operatorname{per} A - p_m(1)| \leq \epsilon.$$

It remains to show that

$$p_m(1) = f(0) + \sum_{k=1}^{m} \frac{f^{(k)}(0)}{k!}$$

is a polynomial of degree at most m in the entries a_{ij} of the matrix A that can be computed in $n^{O(m)}$ time.

As follows from Sect. 2.2.2, the derivatives $f^{(k)}(0)$ for $k = 1, \ldots, m$ can be found in $O(m^2)$ time as linear combinations of the derivatives $g^{(k)}(0)$ for $k = 1, \ldots, m$ with coefficients depending on k only.

For a univariate polynomial $q(z)$ and a positive integer m, let $q_{[m]}(z)$ be the truncated polynomial obtained from q by erasing all monomials of degree higher than m.

Since $\phi(0) = 0$, the constant term of $\phi(z)$ is 0 and to compute $g_{[m]}(z)$, we compute the truncated polynomials $\phi_{[m]}(z)$, $r_{[m]}(z)$ and then truncate the composition $r_{[m]}(\phi_{[m]}(z))$ by discarding all terms of degree higher than m. As in Sect. 3.6.7, we observe that the k-th derivative $r^{(k)}(0)$ is a polynomial of degree k in the entries of the matrix A, which can be computed in $n^{O(k)}$ time. Hence $g^{(k)}(0)$ and thus $f^{(k)}(0)$ are polynomials of degree at most k in the entries a_{ij} of the matrix $A = \left(a_{ij}\right)$. The proof now follows. □

3.8 Permanents of α-Conditioned Matrices and Permutations with Few Cycles

Let $A = \left(a_{ij}\right)$ be an $n \times n$ positive matrix which is α-conditioned for some $\alpha \geq 1$, cf. Definition 3.5.5. Let us fix α and let n grow. It turns out that the bulk of the permanent of A is carried by permutations with a small (logarithmic) number of cycles. We interpret permanents as sums over cycle covers, see Sect. 3.1.3.

The following result was proved in [Ba15].

3.8.1 Theorem. *Let $c(\sigma)$ denote the number of cycles of a permutation $\sigma \in S_n$. For an α-conditioned $n \times n$ matrix $A = \left(a_{ij}\right)$, we have*

$$\sum_{\substack{\sigma \in S_n: \\ c(\sigma) < 3\alpha^2 \ln n + 6}} \prod_{i=1}^{n} a_{i\sigma(i)} \ \geq \ \frac{1}{2} \operatorname{per} A.$$

Given a positive matrix $A = \left(a_{ij}\right)$, we consider the symmetric group S_n as a probability space, where

$$\mathbf{Pr}\,(\sigma) = (\operatorname{per} A)^{-1} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \right) \quad \text{for} \quad \sigma \in S_n.$$

3.8.2 Lemma. *Let us define random variables*

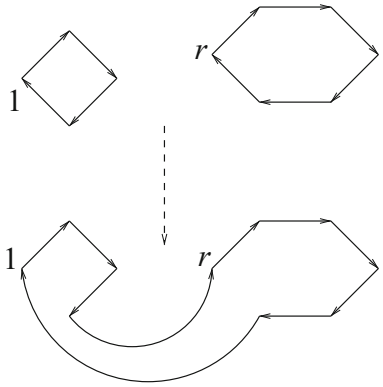
$$l_i : \ S_n \longrightarrow \mathbb{R} \quad \textit{for} \quad i = 1, \ldots, n,$$

where $l_i(\sigma)$ is the length of the cycle of permutation σ that contains i. Assuming that A is α-conditioned, we have

$$\mathbf{Pr}\left(\sigma \in S_n : \ l_i(\sigma) = m\right) \ \leq \ \frac{\alpha^2}{n-m} \quad \textit{for} \quad i = 1, \ldots, n$$

and $m = 1, \ldots, n-1$.

Fig. 3.9 Merging two cycles



Proof. Without loss of generality, we assume that $i = 1$. Let $X \subset S_n$ be the set of permutations $\sigma \in S_n$ such that $l_1(\sigma) = m$. We construct a set $Y \subset S_n$ as follows. Each permutation $\sigma \in X$ contributes $n - m$ permutations into Y: we write the cycle of σ containing 1 as

$$1 = j_1 \to j_2 \to \ldots \to j_m \to 1, \tag{3.8.2.1}$$

pick an element r of the $n - m$ elements not in the cycle, write the cycle of σ containing r as

$$r = j_{m+1} \to j_{m+2} \to \ldots \to j_{m+k} \to r \tag{3.8.2.2}$$

and produce a permutation $\tau \in Y$ by merging the two cycles together:

$$1 = j_1 \to j_2 \to \ldots \to j_m \to r = j_{m+1} \to j_{m+2} \to \ldots \to j_{m+k} \to 1, \tag{3.8.2.3}$$

see Fig. 3.9.

Since A is α-conditioned, we have

$$\mathbf{Pr}\,(\sigma) \;\leq\; \alpha^2 \mathbf{Pr}\,(\tau). \tag{3.8.2.4}$$

Next, we observe that each permutation $\tau \in Y$ is obtained from a unique permutation $\sigma \in X$. To reconstruct σ from τ, we find the cycle of σ containing 1, write it as in (3.8.2.3) and cut into the cycles (3.8.2.1) and (3.8.2.2), see Fig. 3.10

Using (3.8.2.4), we conclude that

$$\mathbf{Pr}\,(X) \;\leq\; \frac{\alpha^2}{n-m} \mathbf{Pr}\,(Y) \;\leq\; \frac{\alpha^2}{n-m}.$$

□

3.8.3 Proof of Theorem 3.8.1. Let l_i be the random variables of Lemma 3.8.2. Using Lemma 3.8.2, we estimate

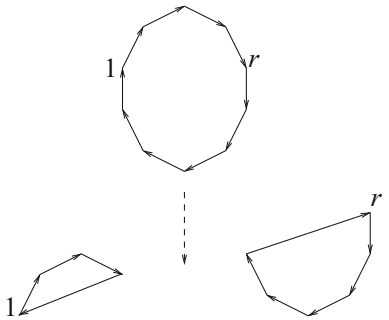


Fig. 3.10 Cutting a cycle into two

$$\begin{aligned}
\mathbf{E}\left(l_i^{-1}\right) &= \sum_{m=1}^{n} \frac{1}{m}\mathbf{Pr}\left(\sigma:\ l_i(\sigma)=m\right)\\
&= \sum_{1\le m\le n/3} \frac{1}{m}\mathbf{Pr}\left(\sigma:\ l_i(\sigma)=m\right) + \sum_{n/3<m\le n} \frac{1}{m}\mathbf{Pr}\left(\sigma:\ l_i(\sigma)=m\right)\\
&\le \frac{3\alpha^2}{2n}\sum_{1\le m\le n/3}\frac{1}{m} + \frac{3}{n}\sum_{n/3<m\le n}\mathbf{Pr}\left(\sigma:\ l_i(\sigma)=m\right)\\
&\le \frac{3\alpha^2\ln n}{2n} + \frac{3}{n}.
\end{aligned}$$

Next, we note that

$$c(\sigma) = \sum_{i=1}^{n} l_i^{-1}(\sigma),$$

since the sum of $l_i^{-1}(\sigma)$ for all i in a cycle of σ is 1. Therefore,

$$\mathbf{E}\,c(\sigma) = \sum_{i=1}^{n}\mathbf{E}\left(l_i^{-1}(\sigma)\right) \le \frac{3\alpha^2\ln n}{2} + 3.$$

Applying the Markov inequality, we conclude that

$$\mathbf{Pr}\left(\sigma:\ c(\sigma) \ge 3\alpha^2\ln n + 6\right) \le \frac{1}{2},$$

and the proof follows. □

As is shown in [Ba15], one immediate corollary of Theorem 3.8.1 is that on α-conditioned matrices, the permanent of A and the *Hamiltonian permanent* of A,

$$\operatorname{ham} A = \sum_{\substack{\sigma \in S_n: \\ c(\sigma)=1}} \prod_{i=1}^{n} a_{i\sigma(i)}$$

differ by a factor of $n^{O(\alpha^2 \ln n)}$ (permutations consisting of a single cycle are called *Hamiltonian cycles*). Similarly to the proof of Lemma 3.8.2, the result is obtained by patching a permutation with $O\left(\alpha^2 \ln n\right)$ cycles into a single cycle. Consequently, for α fixed in advance, using the scaling algorithm of Sect. 3.5, we obtain a polynomial time algorithm for computing $\operatorname{ham} A$ within a factor of $n^{O(\alpha^2 \ln n)}$. As is discussed in [Ba15], this allows one to distinguish in polynomial time directed graphs on n vertices that contain many Hamiltonian cycles (at least $\epsilon^n(n-1)!$ for some fixed $\epsilon > 0$) from graphs that are sufficiently far from having a Hamiltonian cycle (need at least ϵn new edges added to acquire one). The algorithm is obtained by approximating $\operatorname{per} A$ and hence $\operatorname{ham} A$ for a "soft" version $A = \left(a_{ij}\right)$ of the adjacency matrix of the graph,

$$a_{ij} = \begin{cases} 1 & \text{if } i \to j \text{ is an edge} \\ \delta & \text{otherwise} \end{cases}$$

for a sufficiently small $\delta = \delta(\epsilon) > 0$.

Vishnoi [Vi12] used the van der Waerden bound for the permanent (see Sect. 3.3) to prove the existence of long cycles (and of an efficient algorithm to find such cycles) in regular graphs.

3.9 Concluding Remarks

3.9.1 Permanents and determinants. It is tempting to compare the permanent

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

with the syntactically similar determinant

$$\det A = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

and try exploit the similarity. Godsil and Gutman [GG78] suggested the following construction.

Suppose that $A = \left(a_{ij}\right)$ is an $n \times n$ non-negative real matrix. Let ξ_{ij} be real-valued independent random variables such that

$$\mathbf{E}\, \xi_{ij} = 0 \quad \text{and} \quad \mathbf{var}\, \xi_{ij} = 1 \quad \text{for all} \quad i, j = 1, \ldots, n$$

and let us define a random $n \times n$ matrix $B = \left(b_{ij}\right)$ by

$$b_{ij} = \xi_{ij}\sqrt{a_{ij}} \quad \text{for all} \quad i, j = 1, \ldots, n.$$

It is not hard to show that

$$\mathbf{E}\ (\det B)^2 = \operatorname{per} A$$

and one can ask how well $\det^2 B$ is likely to approximate per A, see also Chap. 8 of [LP09]. Since $\det^2 B$ is non-negative, the Markov inequality implies that $\det^2 B$ is unlikely to overestimate per A by a lot (for example, the probability that $\det^2 B > 10 \operatorname{per} A$ does not exceed $1/10$). However, it may happen that $\det^2 B$ grossly underestimates per A. For example, if $n = 2m$ and A is a block-diagonal matrix consisting of m blocks $J_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ then $\operatorname{per} A = 2^m$. If we choose ξ_{ij} to be random signs, so that

$$\mathbf{Pr}\ (\xi_{ij} = 1) = \frac{1}{2} \quad \text{and} \quad \mathbf{Pr}\ (\xi_{ij} = -1) = \frac{1}{2}$$

then $\det B = 0$ with probability $1 - 2^{-m}$. This effect can be mitigated if ξ_{ij} are continuous random variables. In [Ba99] it is shown that if ξ_{ij} are standard Gaussian with density

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

then with probability approaching 1 as n grows, we have

$$(\det B)^2 \ \geq\ (0.28)^n \operatorname{per} A \tag{3.9.1.1}$$

(the worst-case scenario is when $A = I_n$, the $n \times n$ identity matrix). It is also shown that if ξ_{ij} are complex Gaussian with density

$$\frac{1}{\pi} e^{-|z|^2} \quad \text{for} \quad z \in \mathbb{C},$$

in which case $\mathbf{E}\ |\det B|^2 = \operatorname{per} A$ then with probability approaching 1 as n grows, we have

$$|\det B|^2 \ \geq\ (0.56)^n \operatorname{per} A \tag{3.9.1.2}$$

(again, the worst case scenario is when $A = I_n$).

Finally, let us choose ξ_{ij} to be quarternionic Gaussian with density

$$\frac{4}{\pi^2} e^{-|h|^2} \quad \text{for} \quad h \in \mathbb{H}$$

(so that $\mathbf{E}\ |h|^2 = 1$, here $\mathbb{H}$ denotes the skew field of quaternions and not the upper half-plane of $\mathbb{C}$ as elsewhere in the book). Then B is an $n \times n$ quaternionic matrix

which we write as

$$B = R + \mathbf{i}\, S + \mathbf{j}\, T + \mathbf{k}\, U,$$

where R, S, T and U are $n \times n$ real matrices. Let $B_{\mathbb{C}}$ denote the $2n \times 2n$ complex matrix

$$B_{\mathbb{C}} = \begin{pmatrix} R + iS & T + iU \\ -T + iU & R - iS \end{pmatrix}.$$

It is show in [Ba99] that $\det B_{\mathbb{C}}$ is a non-negative real number such that $\mathbf{E}\ \det B_{\mathbb{C}} = \operatorname{per} A$ and that

$$\det B_{\mathbb{C}} \ \geq\ (0.76)^n \operatorname{per} A \tag{3.9.1.3}$$

with probability approaching 1 as n grows (again, the worst-case scenario is when $A = I_n$).

The idea behind the inequalities of (3.9.1.1)–(3.9.1.3) is roughly as follows. We note that $\det B$ is linear in every row of B. We consider $\det B$ as a function of n independent Gaussian n-vectors $x_i = (\xi_{i1}, \ldots, \xi_{in})$. In the real case $(\det B)^2$ is a quadratic form in each x_i, once the values of the remaining vectors $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ are fixed. In the complex case, $|\det B|^2$ is a Hermitian form in each x_i, once the values of the remaining vectors $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ are fixed. In the quaternionic case, $\det B_{\mathbb{C}}$ is a quaternionic Hermitian form in each x_i, once the values of the remaining vectors $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ are fixed.

We deduce (3.9.1.1) from the following: if $q : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a positive semidefinite quadratic form on the space $\mathbb{R}^n$ equipped with the standard Gaussian measure and such that $\mathbf{E}\, q = 1$ then

$$\mathbf{E}\ \ln q \ \geq\ -\ln 2 - \gamma, \tag{3.9.1.4}$$

where $\gamma \approx 0.5772156649$ is the Euler constant and the bound (3.9.1.4) is attained if q is a form of rank 1, for example,

$$q\,(x_1, \ldots, x_n) = x_1^2 \quad \text{where} \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Since every positive semidefinite quadratic form is a convex combination of positive semidefinite forms of rank 1, by Jensen's inequality the minimum in (3.9.1.4) is indeed attained on forms of rank 1. The constant in (3.9.1.1) is $e^{-\ln 2 - \gamma} \approx 0.28$.

We deduce (3.9.1.2) from the following: if $q : \mathbb{C}^n \longrightarrow \mathbb{R}$ is a positive semidefinite Hermitian form on the space $\mathbb{C}^n$ equipped with the standard Gaussian measure and such that $\mathbf{E}\, q = 1$ then

$$\mathbf{E}\ \ln q \ \geq\ -\gamma, \tag{3.9.1.5}$$

and the bound in (3.9.1.5) is attained if q is a form of rank 1, for example,

$$q\,(z_1, \ldots, z_n) = |z_1|^2 \quad \text{where} \quad (z_1, \ldots, z_n) \in \mathbb{C}^n.$$

Similarly to the real case, since every positive semidefinite Hermitian form is a convex combination of positive semidefinite Hermitian forms of rank 1, by Jensen's inequality the minimum in (3.9.1.5) is indeed attained on forms of rank 1. We get a better bound than in the real case, because a complex Hermitian form of rank 1 can be viewed as a real quadratic form of rank 2. The constant in (3.9.1.2) is $e^{-\gamma} \approx 0.56$.

We deduce (3.9.1.3) from the following: if $q : \mathbb{H}^n \longrightarrow \mathbb{R}$ is a positive semidefinite Hermitian form on the space $\mathbb{H}^n$ equipped with the standard Gaussian measure and such that $\mathbf{E}\, q = 1$ then

$$\mathbf{E}\, \ln q \;\geq\; 1 - \gamma - \ln 2 \tag{3.9.1.6}$$

and the bound in (3.9.1.6) is attained if q is a form of rank 1, for example,

$$q\,(h_1, \ldots, h_n) = |h_1|^2 \quad \text{where} \quad (h_1, \ldots, h_n) \in \mathbb{H}^n.$$

The constant in (3.9.1.3) is $e^{1-\gamma-\ln 2} \approx 0.76$.

For various special classes of matrices, a subexponential approximation factor is achieved by (real) Gaussian [F+04], [RZ16] and some non-Gaussian [CV09] random variables ξ_{ij}.

3.9.2 Algorithms for computing permanents. For a general $n \times n$ real or complex matrix A, the most efficient method known of computing per A exactly, is, apparently, Ryser's method and its modifications, see Chap. 7 of [Mi78], which achieves $O(n^2 2^n)$ complexity. Essentially, it uses the formula

$$\operatorname{per} A = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(x_1, \ldots, x_n) \quad \text{where} \quad p(x_1, \ldots, x_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right),$$

and computes the derivative as

$$\frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(x_1, \ldots, x_n) = \sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|} p(x_I), \tag{3.9.2.1}$$

where x_I is the 0–1 vector with 0s in positions I and 1s elsewhere (as is easy to see, formula (3.9.2.1) holds for any homogeneous polynomial p of degree n in $x_1, \ldots, x_n$). The exact computation of the permanent is a **#P**-hard problem already for 0–1 matrices [Va79], which makes a polynomial time algorithm rather unlikely. Efficient (polynomial time) algorithms for computing permanents exactly are known for some rather restricted classes of matrices, for example, for matrices of a small (fixed in advance) rank [Ba96] and for 0–1 matrices with small (fixed in advance) permanents [GK87].

Given an $n \times n$ matrix $A = \big(a_{ij}\big)$, let $G(A)$ be the bipartite graph with $2n$ vertices $1_L, \ldots n_L$ and $1_R, \ldots, n_R$, where vertices i_L and j_R are connected by an edge if and only if $a_{ij} \neq 0$, see Sect. 3.1.2. Cifuentes and Parillo found a polynomial time

algorithm to compute per A exactly provided the treewidth of $G(A)$ is bounded by a constant, fixed in advance [CP16]. The algorithm is applicable to matrices over any commutative ring. One can obtain graphs $G(A)$ of a small treewidth provided A is sufficiently sparse, that is, contains relatively few non-zeros. This is the case, for example, if A has a band structure, that is, $a_{ij} = 0$ provided $|i - j| \geq \omega$ for some ω, fixed in advance.

The greatest success in approximation algorithms is achieved by Jerrum, Sinclair and Vigoda [J+04] who constructed a Markov Chain Monte Carlo based fully polynomial time randomized approximation scheme for computing permanents of non-negative matrices. A scaling based deterministic polynomial time algorithm approximating permanents of $n \times n$ non-negative matrices within a factor of e^n is constructed in [L+00], see also Remark 3.5.4. The approximation factor was improved to 2^n [GS14] and it is conjectured that the same algorithm actually achieves a $2^{n/2}$ approximation factor, cf. (3.4.6.1). Using the "correlation decay" idea from statistical physics, Gamarnik and Katz obtained a $(1 + \epsilon)^n$ approximation factor for any $\epsilon > 0$, fixed in advance, when A is a 0–1 matrix of a constant degree expander graph [GK10].

Less is known about approximation algorithms for not necessarily non-negative matrices (but see Sects. 3.6, 5.5 and also [Mc14]). Gurvits [Gu05] presented a randomized algorithm, which, given an $n \times n$ complex matrix A approximates per A in $O(n^2/\epsilon^2)$ time within an additive error of $\epsilon \|A\|^n$, where $\|A\|$ is the operator norm of A, see also [AA13] for an exposition. The idea of the algorithm is to use the formula

$$\operatorname{per} A = \mathbf{E}\, x_1 \cdots x_n \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right),$$

where $x_i = \pm 1$ are independent Bernoulli random variables and replace the expectation by the sample average.

Chapter 4
Hafnians and Multidimensional Permanents

We explore certain extensions of the permanent: hafnians enumerate perfect matchings in general graphs and multidimensional permanents enumerate perfect matchings in hypergraphs. With the notable exception of the mixed discriminant, which can be thought of as a "permanent-determinant" of a 3-dimensional array, these extensions no longer have connections to $\mathbb{H}$-stable polynomials, which is a major disadvantage. However, other methods we tried on permanents generally continue to work. Using scaling, we establish a decomposition of hafnians and multidimensional permanents into the product of an easy to handle "scaling part" and hard to handle "d-stochastic part". We prove that the d-stochastic part is still concentrated, though weaker than in the case of the permanent. Taylor polynomial interpolation works for hafnians just as well as for permanents, while for multidimensional permanents it produces efficient approximations in non-trivial real and complex domains. The van der Waerden lower bound for mixed discriminants works just as well as for permanents, while for the Bregman - Minc bound, we only manage to obtain a somewhat weaker version.

4.1 Hafnians

4.1.1 Definition. Let $n = 2m$ be a positive even integer and let $A = \big(a_{ij}\big)$ be an $n \times n$ symmetric real or complex matrix. The *hafnian* of A is defined as

$$\operatorname{haf} A = \sum_{\{i_1,i_2\},\ldots,\{i_{2m-1},i_{2m}\}} a_{i_1 i_2} \cdots a_{i_{2m-1} i_{2m}}, \tag{4.1.1.1}$$

where the sum is taken over all $(2m)!/2^m m!$ unordered partitions of the set $\{1, \ldots, n\}$ into unordered pairs (the name was introduced by physicist Eduardo R. Caianiello to mark his fruitful research stay in Copenhagen, or "Hafnia" in Latin).

A. Barvinok, *Combinatorics and Complexity of Partition Functions*,
Algorithms and Combinatorics 30, DOI 10.1007/978-3-319-51829-9_4

Note that the diagonal entries of A are not involved at all. Equivalently,

$$\text{haf } A = \frac{1}{m!2^m} \sum_{\sigma \in S_n} \prod_{i=1}^{m} a_{\sigma(2i-1)\sigma(2i)}, \tag{4.1.1.2}$$

where S_n is the symmetric group of all $n!$ permutations of the set $\{1, \ldots, n\}$. Although one can define the hafnian of any (not necessarily symmetric) matrix by (4.1.1.2), this does not lead to any more generality, since for a skew-symmetric matrix A the expression (4.1.1.2) is identically 0, and, moreover, for a general A the value of (4.1.1.2) is equal to its value on the symmetric part $(A + A^T)/2$ of A.

The permanent of any $m \times m$ matrix is expressed as the hafnian of a $(2m) \times (2m)$ symmetric matrix:

$$\text{per } B = \text{haf } A \quad \text{where} \quad A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.$$

Indeed, any permutation $\sigma \in S_m$ corresponds to the partition τ of $\{1, \ldots, 2m\}$ into pairs $\{i, \sigma(i) + m\}$ for $i = 1, \ldots, m$ and the contributions of σ to per B via (3.1.1.1) and of τ to haf A via (4.1.1.1) coincide. Moreover, any partition τ with a non-zero contribution to haf A corresponds to a unique permutation $\sigma \in S_m$.

We note a recursive formula

$$\text{haf } A = \sum_{j=2}^{n} a_{1j} \text{haf } A_j, \tag{4.1.1.3}$$

where A_j is the $(n-2) \times (n-2)$ symmetric matrix obtained from A by crossing out the first and the j-th row and the first and the j-th column.

4.1.2 Hafnians and perfect matchings. If $A = (a_{ij})$ is a real symmetric matrix and $a_{ij} \in \{0, 1\}$ for all i, j then haf A has a combinatorial interpretation as the number of perfect matchings in the graph G with adjacency matrix A, cf. Sect. 3.1.2. That is, if $G = (V, E)$ is an (undirected, without loops or multiple edges) graph with set $V = \{1, \ldots, n\}$ of vertices and set $E \subset \binom{V}{2}$ of edges, the *adjacency matrix* $A = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Assuming that $n = 2m$ is even, we conclude haf A is the number of perfect matchings of G.

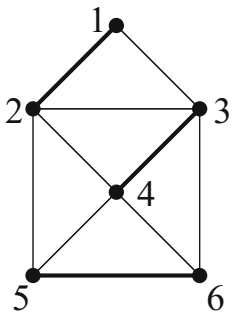


Fig. 4.1 A graph and a perfect matching (thick edges)

For example, Fig. 4.1 pictures a graph with adjacency matrix

$$A = \begin{pmatrix} 0&1&1&0&0&0 \\ 1&0&1&1&1&0 \\ 1&1&0&1&0&1 \\ 0&1&1&0&1&1 \\ 0&1&0&1&0&1 \\ 0&0&1&1&1&0 \end{pmatrix}$$

and a perfect matching of G.

4.1.3 Hafnians as integrals. Let γ_d be the standard Gaussian probability measure on $\mathbb{R}^d$ with density

$$\frac{1}{(2\pi)^{d/2}} e^{-\|x\|^2/2} \quad \text{where} \quad \|x\| = x_1^2 + \ldots + x_d^2 \quad \text{for} \quad x = (x_1, \ldots, x_d)\,.$$

In particular,

$$\mathbf{E}\, x_i^2 = 1 \quad \text{and} \quad \mathbf{E}\, x_i x_j = 0 \quad \text{provided} \quad i \neq j.$$

Let $f_1, \ldots, f_n : \mathbb{R}^d \longrightarrow \mathbb{R}$ be linear forms. Clearly,

$$\mathbf{E}\, f_1 \cdots f_n = 0 \quad \text{if} \quad n \quad \text{is odd.}$$

If $n = 2m$ is even, the expectation of the product is expressed as a hafnian. Namely, let $A = \big(a_{ij}\big)$ be the (necessarily symmetric) $n \times n$ matrix defined by

$$a_{ij} = \mathbf{E}\, f_i f_j = \int_{\mathbb{R}^d} f_i(x) f_j(x)\; d\gamma_d(x).$$

Then

$$\mathbf{E}\, f_1 \cdots f_n = \operatorname{haf}\, A. \tag{4.1.3.1}$$

Formula (4.1.3.1) is known as *Wick's formula*, see [Zv97]. It can be proved as follows. Let us denote the left hand side of (4.1.3.1) by $L(f_1, \ldots, f_n)$ and the right hand side of (4.1.3.1) by $R(f_1, \ldots, f_n)$. For real parameters $t = (t_1, \ldots, t_n)$, let us define

$$f_t = t_1 f_1 + \ldots + t_n f_n.$$

Since $L(f_1, \ldots, f_n)$ and $R(f_1, \ldots, f_n)$ are degree n symmetric multilinear functions of $f_1, \ldots, f_n$, we have

$$L(f_1, \ldots, f_n) = \frac{1}{n!} \frac{\partial^n}{\partial t_1 \cdots \partial t_n} L(f_t, \ldots, f_t) \quad \text{and}$$
$$R(f_1, \ldots, f_n) = \frac{1}{n!} \frac{\partial^n}{\partial t_1 \cdots \partial t_n} R(f_t, \ldots, f_t).$$

Therefore, it suffices to prove (4.1.3.1) assuming that $f_1 = \ldots = f_n$. By the rotational invariance of the measure γ_d, it further suffices to prove (4.1.3.1) when $f_1 = \ldots = f_n$ is the coordinate function, say, x_1. In that case, the matrix A is filled by 1 s and hence the right hand side is equal to

$$\frac{(2m)!}{2^m m!}.$$

The left hand side is

$$\begin{aligned}
\int_{\mathbb{R}^d} x_1^{2m} \, d\gamma_d(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^{2m} e^{-x^2/2} \, dx = \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} (2t)^m e^{-t} \, \frac{dt}{\sqrt{2t}} \\
&= \frac{1}{\sqrt{\pi}} 2^m \int_0^{+\infty} t^{m-\frac{1}{2}} e^{-t} \, dt = \frac{2^m}{\sqrt{\pi}} \Gamma\left(m + \frac{1}{2}\right) \\
&= \frac{2^m}{\sqrt{\pi}} \left(m - \frac{1}{2}\right)\left(m - \frac{3}{2}\right) \cdots \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \\
&= (2m-1)(2m-3) \cdots 1 = \frac{(2m-1)!}{(2m-2) \cdots (2m-4) \cdots 2} \\
&= \frac{(2m-1)!}{2^{m-1}(m-1)!} = \frac{(2m)!}{2^m m!},
\end{aligned}$$

which completes the proof of (4.1.3.1).

One corollary of (4.1.3.1) is that if A is an $n \times n$ symmetric positive semidefinite matrix then

$$\operatorname{haf} B \geq 0 \quad \text{for} \quad B = \begin{pmatrix} A & A \\ A & A \end{pmatrix}.$$

Indeed, $A = \left(a_{ij}\right)$ can be written as

$$a_{ij} = \mathbf{E} \, f_i f_j \quad \text{for all} \quad i, j$$

and some linear forms $f_1, \ldots, f_n : \mathbb{R}^n \longrightarrow \mathbb{R}$, in which case

$$\text{haf } B = \mathbf{E}\, f_1^2 \cdots f_n^2 \;\geq\; 0.$$

The following useful inequality relates hafnians and permanents of non-negative matrices.

4.1.4 Theorem. *Let A be an $n \times n$ non-negative symmetric matrix, where n is even. Then*

$$\text{haf } A \;\leq\; \sqrt{\text{per } A}.$$

Proof. We follow [AF08]. Let $n = 2m$ and let us consider $(\text{haf } A)^2$ as a polynomial in the entries a_{ij} of the matrix A.

From the definition (4.1.1.1), we can write

$$(\text{haf } A)^2 = \sum_{I,J} a_{i_1 i_2} \cdots a_{i_{2m-1} i_{2m}} a_{j_1 j_2} \cdots a_{j_{2m-1} j_{2m}}, \tag{4.1.4.1}$$

where the sum is taken over all ordered pairs (I, J) of unordered partitions of the set $\{1, \ldots, 2m\}$ into unordered pairs $I = \big\{\{i_1, i_2\}, \ldots, \{i_{2m-1}, i_{2m}\}\big\}$ and $J = \big\{\{j_1, j_2\}, \ldots, \{j_{2m-1}, j_{2m}\}\big\}$ (we allow $I = J$ and count such pairs once). For given I and J, the union of all pairs in I and J can be viewed as a graph with set $\{1, \ldots, n\}$ of vertices and possibly multiple edges such that each vertex belongs to exactly two edges, counting multiplicities, see Fig. 4.2. Such a graph is a union of disjoint cycles, each cycle consisting of an even number of edges (counting multiplicities). On the other hand, let Γ be a graph which is a union of disjoint cycles, each consisting of an even number of edges, possibly including cycles with two edges, and containing all n vertices. Let $c_{>2}(\Gamma)$ be the number of cycles of Γ with more than 2 edges. Then Γ

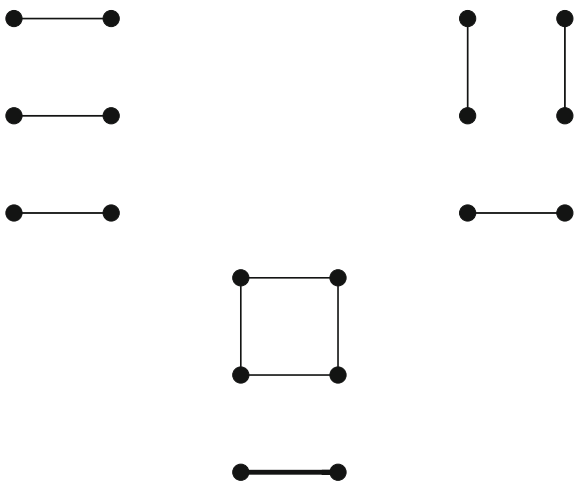


Fig. 4.2 Two matchings and their union

can be represented as a union of two perfect matchings in exactly $2^{c_{>2}(\Gamma)}$ ways and hence (4.1.4.1) can be written as

$$(\text{haf } A)^2 = \sum_{\substack{\Gamma: \\ \text{each cycle has even length}}} 2^{c_{>2}(\Gamma)} \prod_{\{i,j\}\in\Gamma} a_{ij}. \tag{4.1.4.2}$$

To obtain the monomial expansion of per A, we interpret A as the adjacency matrix of a complete directed graph on n vertices, which includes loops $i \to i$ and edges in both directions $i \to j$ and $j \to i$ for $i \neq j$, see Sect. 3.1.3. Then

$$\text{per } A = \sum_{\vec{\Gamma}} \prod_{(i,j)\in\vec{\Gamma}} a_{ij}, \tag{4.1.4.3}$$

where the sum is taken over all directed cycle covers $\vec{\Gamma}$ of the complete graph. Since A is symmetric, the contributions of any two $\vec{\Gamma}_1$ and $\vec{\Gamma}_2$ that differ just by orientations on their cycles are the same and therefore (4.1.4.3) can be written as

$$\text{per } A = \sum_{\Gamma} 2^{c_{>2}(\Gamma)} \prod_{\{i,j\}\in\Gamma} a_{ij}, \tag{4.1.4.4}$$

where the sum is taken over all graphs Γ that are disjoint union of undirected cycles and contain all vertices $\{1, \ldots, n\}$ and where $c_{>2}(\Gamma)$ is the number of cycles in Γ consisting of more than 2 edges. Comparing (4.1.4.2) and (4.1.4.4), we conclude that

$$\text{per } A \;\geq\; (\text{haf } A)^2.$$

□

The results of Sects. 3.6 and 3.7 almost verbatim transfer from permanents to hafnians.

4.1.5 Theorem. *There exists an absolute constant $\delta_0 > 0$ (one can choose $\delta_0 = 0.5$) such that for any even integer n and for any $n \times n$ symmetric matrix $A = (a_{ij})$ with complex entries satisfying*

$$|1 - a_{ij}| \;\leq\; \delta_0 \quad \textit{for all} \quad i \neq j$$

we have

$$\text{haf } A \neq 0.$$

For any $0 < \delta < \delta_0$ there exists $\gamma = \gamma(\delta) > 0$ and for any $0 < \epsilon < 1$ and positive even integer n there exists a polynomial $p = p_{n,\delta,\epsilon}$ in the entries of an $n \times n$ complex symmetric matrix $A = (a_{ij})$ satisfying

$$\deg p \;\leq\; \gamma(\ln n - \ln \epsilon)$$

and such that

$$|\ln \operatorname{haf}\ A - p(A)| \ \leq\ \epsilon$$

provided

$$\left|1 - a_{ij}\right| \ \leq\ \delta \quad \textit{for all} \quad i \neq j.$$

Proof. The proof closely follows those of Theorems 3.6.1 and 3.6.2. First, we show by induction on m that if $A = \left(a_{ij}\right)$ and $B = \left(b_{ij}\right)$ are two symmetric $(2m) \times (2m)$ complex matrices satisfying

$$\left|1 - a_{ij}\right| \ \leq\ 0.5 \quad \text{and} \quad \left|1 - b_{ij}\right| \ \leq\ 0.5 \qquad \text{for all} \qquad i \neq j$$

and such that the entries of A and B coincide except possibly in the i-th row and i-th column for some unique i then $\operatorname{haf}\ A \neq 0$, $\operatorname{haf}\ B \neq 0$ and the angle between non-zero complex numbers $\operatorname{haf}\ A$ and $\operatorname{haf}\ B$ does not exceed $\pi/2$.

This clearly holds for $m = 1$. Assuming that $m > 1$, without loss of generality we assume that B is obtained from A by replacing the entries $a_{1j} = a_{j1}$ by $b_{1j} = b_{j1}$ for $j = 2, \ldots, 2m$. Using (4.1.1.3), we write

$$\operatorname{haf}\ A = \sum_{j=2}^{2m} a_{1j} \operatorname{haf}\ A_j \quad \text{and} \quad \operatorname{haf}\ B = \sum_{j=2}^{2m} b_{1j} \operatorname{haf}\ A_j,$$

where A_j is the $(2m-2) \times (2m-2)$ matrix obtained from A by crossing out the first and the j-th row and the first and the j-th column. We note that, up to a simultaneous permutation of rows and columns, any two matrices A_{j_1} and A_{j_2} differ in at most the i-th row and i-th column for some unique i, so by the induction hypothesis $\operatorname{haf}\ A_j \neq 0$ for all $j = 2, \ldots, 2m$ and the angle between any two non-zero complex numbers $\operatorname{haf}\ A_{j_1}$ and $\operatorname{haf}\ A_{j_2}$ does not exceed $\pi/2$. Applying Lemma 3.6.4 with $u_j = \operatorname{haf}\ A_j$, $a_j = a_{1j}$ and $b_j = b_{1j}$, as in Sect. 3.6.5, we conclude that $\operatorname{haf}\ A \neq 0$, $\operatorname{haf}\ B \neq 0$ and the angle between $\operatorname{haf}\ A$ and $\operatorname{haf}\ B$ does not exceed $\pi/2$.

Next, we construct the polynomial p. Let $J = J_n$ be the $n \times n$ matrix filled with 1 s and let $n = 2m$. We define the polynomial

$$g(z) = \operatorname{haf}\ (J + z(A - J))$$

of degree at most m, so that

$$g(0) = \operatorname{haf}\ J = \frac{(2m)!}{2^m m!} \quad \text{and} \quad g(1) = \operatorname{haf}\ A.$$

Moreover, for $\beta = \delta_0/\delta > 1$, we have $g(z) \neq 0$ whenever $|z| \leq \beta$. We choose a branch of $f(z) = \ln g(z)$ for $|z| \leq 1$ such that $f(0)$ is real and use Lemma 2.2.1 to claim that for some $k \leq \gamma(\ln n - \ln \epsilon)$ the Taylor polynomial

$$p_k(z) = f(0) + \sum_{s=1}^{k} \frac{f^{(s)}(0)}{s!} z^s$$

approximates $f(z)$ for $|z| \leq 1$ within an additive error ϵ. We need to show that $p_k(1)$ is a polynomial of degree at most k in the entries of A. To finish the proof as in Sect. 3.6.7, it suffices to show that $g^{(s)}(0)$ is a polynomial in the entries of A of degree at most s.

Indeed,

$$\frac{d^s}{dz^s} g(z)\Big|_{z=0} = \frac{d^s}{dz^s} \sum_{\{i_1,j_1\},\dots,\{i_m,j_m\}} \left(1 + z\left(a_{i_1 j_1} - 1\right)\right) \cdots \left(1 + z\left(a_{i_m,j_m} - 1\right)\right)\Big|_{z=0},$$

where the sum is taken over all unordered partitions of the set $\{1, \dots, n\}$ into m unordered pairs $\{i_1, j_1\}, \dots, \{i_m, j_m\}$. Therefore,

$$g^{(s)}(0) = \frac{(2m-2s)!s!}{(m-s)!2^{m-s}} \sum_{\{i_1,j_1\},\dots,\{i_s,j_s\}} \left(a_{i_1 j_1} - 1\right) \cdots \left(a_{i_m,j_m} - 1\right),$$

where the sum i of s pairwise disjoint unordered pairs $\{i_1, j_1\}, \dots, \{i_s, j_s\}$. □

We observe that for a fixed $\delta < \delta_0$, the polynomial $p(A)$ in Theorem 4.1.5 can be computed in $n^{O(\ln n - \ln \epsilon)}$ time.

4.1.6 Theorem. *Let us fix a real* $0 \leq \delta < 1$ *and let*

$$\tau = (1-\delta)\sin\left(\frac{\pi}{4} - \arctan\delta\right) > 0.$$

For an even n, let $Z = \left(z_{ij}\right)$ *be an* $n \times n$ *symmetric complex matrix such that*

$$\left|1 - \Re\, z_{ij}\right| \leq \delta \quad \textit{and} \quad \left|\Im\, z_{ij}\right| \leq \tau \quad \textit{for all} \quad i, j.$$

Then

$$\text{haf } Z \neq 0.$$

As in Sect. 3.7, we deduce from Theorem 4.1.6 the following result.

4.1.7 Theorem. *For any* $0 \leq \delta < 1$ *there exists* $\gamma = \gamma(\delta) > 0$ *such that for any positive even integer n and any real* $0 < \epsilon \leq 1$ *there exists a polynomial* $p = p_{n,\delta,\epsilon}$ *in the entries of an* $n \times n$ *symmetric matrix A such that* $\deg p \leq \gamma(\ln n - \ln \epsilon)$ *and*

$$|\ln \text{haf } A - p(A)| \leq \epsilon$$

provided $A = \left(a_{ij}\right)$ *is a real symmetric matrix satisfying*

$$\left|1 - a_{ij}\right| \leq \delta \quad \textit{for all} \quad i, j.$$

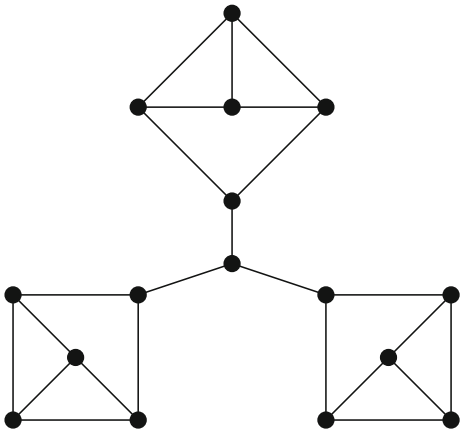


Fig. 4.3 A connected 3-regular graph with no perfect matchings

Similarly, for any $\delta > 0$, fixed in advance, the polynomial p can be computed in $n^{O(\ln n - \ln \epsilon)}$ time.

The proofs of Theorems 4.1.6 and 4.1.7 closely follow the proofs of Sect. 3.7 with necessary adjustments as in the proof of Theorem 4.1.5, see also [B16+].

The main difficulty of dealing with hafnians compared to dealing with permanents is that there appears to be no parallel theory relating hafnians to stable polynomials, cf. Sects. 3.2–3.3, but see also Sect. 6 of [FG06] for an attempt to extend the theory to hafnians. Consequently, there is no analogue of the van der Waerden inequality (Theorem 3.3.2) for hafnians. As the following simple example shows, the hafnian of a symmetric doubly stochastic matrix can be equal to 0. Indeed, if G is a graph that is a disjoint union of an even number of triangles, and A is the adjacency matrix of G then $B = (1/2)A$ is a symmetric doubly stochastic matrix and $\operatorname{haf} B = 0$.

Figure 4.3 demonstrates a more complicated example of a 3-regular graph without perfect matchings.

If A is the adjacency matrix of the graph on Fig. 4.3, then $B = (1/3)A$ is a symmetric doubly stochastic matrix and $\operatorname{haf} B = 0$. On the other hand, the number of perfect matchings in a bridgeless 3-regular graph is exponentially large in the number of vertices [E+11].

4.2 Concentration of Hafnians of α-Conditioned Doubly Stochastic Matrices

Although there is no hafnian analogue of the van der Waerden inequality, some of the corollaries of that inequality can be extended to hafnians, in particular, concentration of hafnians of doubly stochastic matrices with relatively uniform entries, see Sect. 3.4.6. We start with a definition.

4.2.1 Definition. Let $A = (a_{ij})$ be a symmetric matrix with zero diagonal and positive off-diagonal entries. For $\alpha \geq 1$, we say that A is *α-conditioned* if

$$a_{ij_1} \leq \alpha a_{ij_2} \quad \text{for all} \quad i \neq j_1, j_2.$$

The goal of this section is to prove the following result.

4.2.2 Theorem. *For any $\alpha \geq 1$, there is a $\gamma = \gamma(\alpha) > 0$ such that if A is a $2m \times 2m$ symmetric doubly stochastic α-conditioned matrix with zero diagonal, we have*

$$m^{-\gamma} e^{-m} \leq \operatorname{haf} A \leq m^{\gamma} e^{-m}.$$

We follow [BS11]. First, we need to adapt the technique of matrix scaling, see Sect. 3.5, to hafnians.

4.2.3 Scaling. Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix with zero diagonal. We say that A is obtained by *scaling* from an $n \times n$ symmetric matrix $B = (b_{ij})$ if

$$a_{ij} = \lambda_i \lambda_j b_{ij} \quad \text{for all} \quad i, j$$

and some $\lambda_1, \ldots, \lambda_n$. If n is even, then the hafnians of A and B are defined and

$$\operatorname{haf} A = \left(\prod_{i=1}^{n} \lambda_i \right) \operatorname{haf} B.$$

Note that compared to scaling of general matrices, we get just n scaling factors λ_i, instead of $2n$ factors λ_i and μ_j in the case of the permanent.

The following result is a more or less straightforward extension of Theorem 3.5.2 and Lemma 3.5.3.

4.2.4 Theorem. *Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix with zero diagonal and positive off-diagonal entries. Then there exists a unique $n \times n$ symmetric doubly stochastic matrix $B = (b_{ij})$ and unique positive $\lambda_1, \ldots, \lambda_n$ such that*

$$b_{ij} = \lambda_i \lambda_j a_{ij} \quad \textit{for all} \quad i, j.$$

The matrix B can be found as the minimum point of the convex function

$$f(X) = \sum_{1 \leq i \neq j \leq n} x_{ij} \ln \frac{x_{ij}}{a_{ij}}$$

on the polyhedron of $n \times n$ symmetric doubly stochastic matrices X with zero diagonal, in which case

$$f(B) = 2 \sum_{i=1}^{n} \ln \lambda_i.$$

Let $\mathcal{C} = \mathcal{C}(A) \subset \mathbb{R}^n$ be the convex set defined by

$$\mathcal{C} = \left\{ x = (x_1, \ldots, x_n) : \sum_{1 \le i \ne j \le n} a_{ij} e^{x_i + x_j} \le n \right\}$$

and let $x_0 = (\xi_1, \ldots, \xi_n)$ where $\xi_i = \ln \lambda_i$. Then x_0 is the unique maximum point of the linear function $\ell(x) = x_1 + \ldots + x_n$ on $\mathcal{C}$.

Proof. The proof of the first part is very similar to the proof of Theorem 3.5.2 and therefore omitted. To prove the second part, we observe that the point x_0 lies on the boundary of $\partial\mathcal{C}$ that is a smooth strictly convex hypersurface defined by the equation

$$\sum_{1 \le i \ne j \le n} a_{ij} e^{x_i + x_j} = n,$$

cf. Sect. 2.1.1.3. Moreover, the gradient of $g(x) = \sum_{i \ne j} a_{ij} e^{x_i + x_j}$ at x_0 is $(2, \ldots, 2)$, from which it follows that the affine hyperplane H defined by the equation $\ell(x) = \ell(x_0)$ is tangent to $\partial\mathcal{C}$ at x_0. Since $\mathcal{C}$ is convex, H is the supporting affine hyperplane at x_0 and hence x_0 is an extremal point of ℓ. Then x_0 has to be the maximum point, because ℓ is unbounded from below on $\mathcal{C}$. □

Our next result is a version of Lemma 3.5.6 for hafnians.

4.2.5 Lemma. *Let A be an α-conditioned $n \times n$ symmetric matrix with zero diagonal. Suppose that A is obtained by scaling from a doubly stochastic symmetric $n \times n$ matrix B. Then B is α^2-conditioned.*

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the scaling factors so that

$$b_{ij} = \lambda_i \lambda_j a_{ij} \quad \text{for all} \quad i, j.$$

Let us choose two $1 \le i \ne j \le n$. Then

$$\sum_{k \ne i,j} a_{ik} \lambda_i \lambda_k = \sum_{k \ne i,j} b_{ik} = 1 - b_{ij} \tag{4.2.5.1}$$

and

$$\sum_{k \ne i,j} a_{jk} \lambda_j \lambda_k = \sum_{k \ne i,j} b_{jk} = 1 - b_{ij}. \tag{4.2.5.2}$$

Comparing (4.2.5.1) and (4.2.5.2) and using that A is α-conditioned, we conclude that

$$\lambda_i \le \alpha \lambda_j \quad \text{for all} \quad i, j,$$

from which B is α^2-conditioned. □

We prove Theorem 4.2.2 by induction on m, for which we need yet another lemma which bounds the product of the scaling factors of an α-conditioned matrix.

4.2.6 Lemma. *For $n > 2$, let $A = (a_{ij})$ be an α-conditioned $n \times n$ symmetric matrix with zero diagonal. Suppose that*

$$\sum_{1 \le i \ne j \le n} a_{ij} = n$$

and that

$$\left| 1 - \sum_{j : j \ne i} a_{ij} \right| \le \frac{\beta}{n} \quad \text{for} \quad i = 1, \ldots, n$$

and some

$$0 \le \beta \le \frac{n-2}{2\alpha}.$$

Suppose that A is obtained from a symmetric doubly stochastic matrix $B = (b_{ij})$ by scaling, so that

$$b_{ij} = \lambda_i \lambda_j a_{ij} \quad \text{for all} \quad i, j$$

and some positive $\lambda_1, \ldots, \lambda_n$. Then

$$0 \le \sum_{i=1}^{n} \ln \lambda_i \le \frac{8\beta^2\alpha}{n}.$$

Proof. Let us define

$$\delta_i = 1 - \sum_{j : j \ne i} a_{ij} \quad \text{for} \quad i = 1, \ldots, n,$$

so that

$$\sum_{i=1}^{n} \delta_i = 0 \tag{4.2.6.1}$$

and

$$|\delta_i| \le \frac{\beta}{n} \quad \text{for} \quad i = 1, \ldots, n. \tag{4.2.6.2}$$

Let us define an $n \times n$ matrix $X = (x_{ij})$ by

$$x_{ij} = a_{ij} + w_{ij} \quad \text{where} \quad w_{ij} = \frac{\delta_i + \delta_j}{n-2} \quad \text{for} \quad i \ne j.$$

and $x_{ii} = 0$ for $i = 1, \ldots, n$. We observe that X is a symmetric $n \times n$ matrix with row and column sums 1 and zero diagonal. Moreover, since A is α-conditioned, we have

$$a_{ij} \;\geq\; \frac{1}{(n-1)\alpha} \quad \text{for all} \quad i \neq j \tag{4.2.6.3}$$

while by (4.2.6.2), we have

$$\left|w_{ij}\right| \;\leq\; \frac{2\beta}{n(n-2)} \;\leq\; \frac{1}{n\alpha} \tag{4.2.6.4}$$

and hence $x_{ij} \geq 0$, so X is doubly stochastic.

By Theorem 4.2.4,

$$\begin{aligned}\sum_{i=1}^{n} \ln \lambda_i \;&\leq \frac{1}{2} \sum_{1 \leq i \neq j \leq n} x_{ij} \ln \frac{x_{ij}}{a_{ij}} = \frac{1}{2} \sum_{1 \leq i \neq j \leq n} \left(a_{ij} + w_{ij}\right) \ln \frac{a_{ij} + w_{ij}}{a_{ij}} \\ &\leq \frac{1}{2} \sum_{1 \leq i \neq j \leq n} \left(a_{ij} + w_{ij}\right) \frac{w_{ij}}{a_{ij}} = \frac{1}{2} \sum_{1 \leq i \neq j \leq n} \left(w_{ij} + \frac{w_{ij}^2}{a_{ij}}\right).\end{aligned}$$

Now, by (4.2.6.1)

$$\sum_{1 \leq i \neq j \leq n} w_{ij} = \frac{1}{n-2} \sum_{1 \leq i \neq j \leq n} (\delta_i + \delta_j) = 0$$

and by (4.2.6.2)–(4.2.6.4),

$$\sum_{1 \leq i \neq j \leq n} \frac{w_{ij}^2}{a_{ij}} \;\leq\; n(n-1) \frac{4\beta^2 \alpha (n-1)}{n^2 (n-2)^2} \;\leq\; \frac{16 \beta^2 \alpha}{n},$$

which proves the upper bound for $\sum_{i=1}^{n} \ln \lambda_i$. To prove the lower bound, we note that $x = (0, \ldots, 0)$ is a feasible point of the set $\mathcal{C}(A)$ of Theorem 4.2.4 and hence

$$\sum_{i=1}^{n} \ln \lambda_i \;\geq\; 0.$$

□

4.2.7 Proof of Theorem 4.2.2. All implicit constants in the "O" notation below depend only on α.

For a set $I \subset \{1, \ldots, 2m\}$, let $A(I)$ denote the submatrix of A consisting of the entries a_{ij} with $i, j \in I$. Hence $A(I)$ is a symmetric α-conditioned with zero diagonal. Let $B(I)$ be the doubly stochastic matrix obtained from $A(I)$ by scaling. We prove by induction on $k = 1, \ldots, m$ that

$$\text{haf } B(I) = \exp\left\{-k + O\left(\sum_{j=1}^{k} \frac{1}{j}\right)\right\} \quad \text{where} \quad |I| = 2k. \tag{4.2.7.1}$$

Let $I \subset \{1, \ldots, 2m\}$ be a subset such that $|I| = 2k > 2$. Let us pick an $i \in I$. To simplify the notation somewhat, we denote $B(I)$ just by B and assume without loss of generality that $i = 1$. We use the row expansion (4.1.1.3):

$$\text{haf } B = \sum_{j \in I \setminus \{1\}} b_{1j} \text{ haf } B_j, \tag{4.2.7.2}$$

where B_j is the matrix obtained from B by crossing out the 1st and the jth row and the 1st and the jth column. Note that (4.2.7.2) represents haf B as a convex combination of haf B_j.

By Lemma 4.2.5, the matrix B is α^2-conditioned. Since B is doubly stochastic, it follows that the entries of B do not exceed $\alpha^2/(2k-1)$. Let σ_j be the sum of the matrix entries of B_j. Hence

$$\sigma_j = 2k - 4 + O\left(\frac{1}{k}\right). \tag{4.2.7.3}$$

Let us scale B_j to the total sum of entries $2k-2$, so we define

$$\widehat{B}_j = \frac{2k-2}{\sigma_j} B_j \quad \text{for} \quad j \in I \setminus \{1\}.$$

Then

$$\text{haf } B_j = \left(\frac{\sigma_j}{2k-2}\right)^{k-1} \text{haf } \widehat{B}_j \quad \text{for} \quad j \in I \setminus \{1\}$$

and by (4.2.7.3) we conclude that

$$\text{haf } B_j = \exp\left\{-1 + O\left(\frac{1}{k}\right)\right\} \text{haf } \widehat{B}_j. \tag{4.2.7.4}$$

To estimate haf $\widehat{B}_j$, we apply Lemma 4.2.6. Let us scale $\widehat{B}_j$ to a doubly stochastic matrix. The doubly stochastic matrix we get is the same matrix we obtain from $A(I \setminus \{1, j\})$ by scaling, that is, the matrix $B(I \setminus \{1, j\})$.

Since B_j is obtained by crossing out two rows and two columns of a doubly stochastic matrix B, the row and column sums of B_j do not exceed 1, but since the entries of B do not exceed $\alpha^2/(2k-1)$, the row and column sums of B_j are at least

$$1 - \frac{2\alpha^2}{2k-1}.$$

By (4.2.7.3), the absolute value of the difference between any row or column sum of $\widehat{B}_j$ and 1 is $O(1/k)$.

Applying Lemma 4.2.6, we conclude that for all $k \geq \gamma_1(\alpha)$, we have

$$\text{haf } \widehat{B}_j = \exp\left\{O\left(\frac{1}{k}\right)\right\} \text{haf } B\,(I \setminus \{1, j\})\,, \tag{4.2.7.5}$$

where $\gamma_1(\alpha)$ is some positive constant. We use a trivial estimate

$$\text{haf } B = e^{O(1)} \quad \text{provided} \quad k < \gamma_1(\alpha). \tag{4.2.7.6}$$

Combining (4.2.7.6), (4.2.7.5), (4.2.7.2) and the induction hypothesis, we complete the proof of (4.2.7.1). □

4.2.8 Remark. The gist of Theorem 4.2.2 is the lower bound for haf A. As one can see from the proof, we get a much better upper bound combining the inequalities of Theorem 4.1.4 and Corollary 3.4.5.

4.3 Hafnians and Pfaffians

4.3.1 Pfaffian. Let n be a positive even integer, $n = 2m$, and let $A = \left(a_{ij}\right)$ be an $n \times n$ skew-symmetric matrix, so that $a_{ij} = -a_{ji}$ for all $1 \leq i, j \leq n$. The *Pfaffian* of A is defined as

$$\text{Pf } A = \frac{1}{m!2^m} \sum_{\sigma \in S_n} (\text{sgn } \sigma) \prod_{i=1}^{m} a_{\sigma(2i-1)\sigma(2i)}, \tag{4.3.1.1}$$

see, for example, Sect. 1 of Chap. VI of [We97] or Chap. 29 of [Pr94]. Note that while different permutations σ may contribute the same product in (4.3.1.1), all those products are counted with the same sign: if $\sigma_1 = \sigma_2\tau$, where τ is the transposition, $\tau = (2i - 1, 2i)$, say, then $\text{sgn } \sigma_1 = -\text{sgn } \sigma_2$ but since A is skew-symmetric, the signs of monomials in (4.3.1.1) corresponding to σ_1 and σ_2 coincide. Similarly, if $\sigma_1 = \sigma_2\tau$ where τ is the product of two transpositions, $\tau = (2i_1-1, 2i_2-1)(2i_1, 2i_2)$, then $\text{sgn } \sigma_1 = \text{sgn } \sigma_2$ and the signs of monomials corresponding to σ_1 and σ_2 coincide.

One can of course define Pf A for an arbitrary matrix A by (4.3.1.1) but then the Pfaffian of an arbitrary matrix will coincide with the Pfaffian of its skew-symmetric part:

$$\text{Pf } A = \text{Pf}\left(\frac{A - A^T}{2}\right).$$

Assuming that A is a skew-symmetric complex matrix, we may identify with A the exterior 2-form $\omega_A \in \bigwedge^2 \mathbb{C}^n$,

$$\omega_A = \sum_{1 \le i < j \le n} a_{ij} e_i \wedge e_j,$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$. In this case,

$$\underbrace{\omega_A \wedge \ldots \wedge \omega_A}_{m \text{ times}} = (m! \operatorname{Pf} A)\, e_1 \wedge \ldots \wedge e_n. \tag{4.3.1.2}$$

Let G be an $n \times n$ complex matrix. Then the matrix $B = G^T A G$ is skew-symmetric and

$$\omega_B = \sum_{1 \le i < j \le n} a_{ij} (G e_i) \wedge (G e_j).$$

Since

$$(G e_1) \wedge \ldots \wedge (G e_n) = (\det G)\, (e_1 \wedge \ldots \wedge e_n),$$

it follows from (4.3.1.2) that

$$\operatorname{Pf}(G A G^T) = (\det G) \operatorname{Pf} A. \tag{4.3.1.3}$$

Equation (4.3.1.3) allows us to compute $\operatorname{Pf} A$ efficiently: indeed, for every $2m \times 2m$ skew-symmetric matrix A, one can easily compute a matrix G such that $A = G^T K G$, where K is a $2m \times 2m$ block-diagonal matrix with blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so

$$K = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{4.3.1.4}$$

see, for example, Sect. 21 of [Pr94]. Then

$$\operatorname{Pf} A = \operatorname{Pf}\left(G^T K G\right) = (\det G) \operatorname{Pf} K = \det G.$$

4.3.2 Perfect matchings in directed graphs. Let $\vec{H}$ be a directed graph with set $\{1, \ldots, n\}$ of vertices, no loops and at most one edge $\vec{ij}$ or $\vec{ji}$ connecting any two vertices i and j. We assume that n is an even integer, $n = 2m$. A collection $I = \left\{\vec{i_1 i_2}, \ldots, \vec{i_{n-1} i_n}\right\}$ of pairwise disjoint edges of $\vec{H}$ is called a *perfect matching* of $\vec{H}$.

We define sgn $I = \operatorname{sgn} \sigma$ where $\sigma \in S_n$ is a permutation such that $\sigma(k) = i_k$ for $k = 1, \ldots, n$ (as before, the sign of I does not depend on the order in which we list the edges of I).

Let I and J be two perfect matchings in $\vec{H}$ then the union $I \cup J$ is a cycle cover Γ of $\vec{H}$ by even cycles (that is, cycles having an even length), cf. Fig. 4.4.

We call a cycle of Γ *evenly oriented* if, when we choose an orientation of the cycle, the number of edges co-oriented with the cycle is even. Otherwise, we call the cycle *oddly oriented*. Since the cycle is even, the definition does not depend on the choice of an orientation of the cycle. For example, on Fig. 4.4, the 6-cycle is evenly oriented while the 4- and 2-cycles are oddly oriented.

4.3.3 Lemma. *For any two perfect matchings I and J of $\vec{H}$, we have*

$$(\operatorname{sgn} I)(\operatorname{sgn} J) = (-1)^k,$$

where k is the number of evenly oriented cycles in $I \cup J$.

Proof. First, we observe that if the conclusion of the lemma holds for $\vec{H}$, it also holds for the graph obtained from $\vec{H}$ by reversing the direction of one edge. Indeed, if that edge belongs neither to I nor to J then reversing its direction does not change $\operatorname{sgn} I$, $\operatorname{sgn} J$ or k. If the edge belongs to I and to J both, then reversing its direction changes both $\operatorname{sgn} I$ and $\operatorname{sgn} J$. However, since that edge forms a 2-cycle in $I \cup J$, which is always oddly oriented, cf. Fig. 4.4, changing the direction of the edge does not change the number of evenly oriented cycles. Finally, if the edge belongs to I and not to J then changing the direction of the edge reverses $\operatorname{sgn} I$, leaves $\operatorname{sgn} J$ intact and changes k by 1.

Therefore, without loss of generality, we assume that all cycles of length greater than 2 in $I \cup J$ are oriented, cf. Fig. 4.5.

In this case, k is the number of cycles of Γ of length greater than 2. We define two permutations $\sigma, \tau \in S_n$ as follows: We number the cycles of Γ, listing the cycles of length greater than 2 first, list the vertices of the first cycle in the order of the cycle, then the vertices of the second cycle in the order of the cycle, etc., just obtaining a permutation $i_1 \ldots i_n$. We define $\sigma(l) = i_l$ for $l = 1, \ldots, n$. To define $\tau(l)$, we first determine the cycle in which the l-th vertex i_l lies. If i_l lies in a cycle of length greater than 2, we let $\tau(l)$ to be the next vertex of the same cycle in the order of the cycle. If i_l lies in the cycle of length 2, we let $\tau(l) = i_l$.

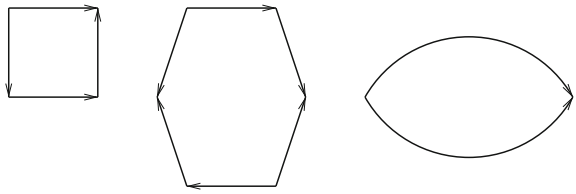


Fig. 4.4 A cycle cover Γ of $\vec{H}$

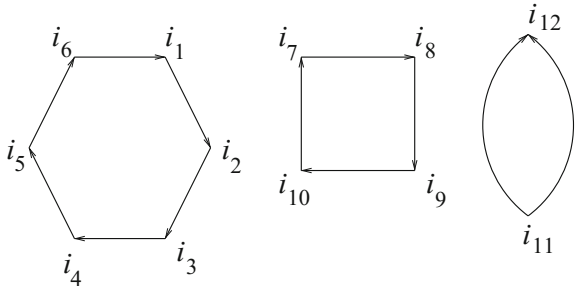


Fig. 4.5 Oriented cycles and a 2-cycle

For example, for the cycle cover on Fig. 4.5, we define $\sigma(l) = i_l$ for $l = 1, \ldots, 12$ and $\tau(1) = i_2$, $\tau(2) = i_3$, $\tau(3) = i_4$, $\tau(4) = i_5$, $\tau(5) = i_6$, $\tau(6) = i_1$, $\tau(7) = i_8$, $\tau(8) = i_9$, $\tau(9) = i_{10}$, $\tau(10) = i_7$, $\tau(11) = i_{11}$ and $\tau(12) = i_{12}$, in which case σ corresponds to the perfect matching $\overrightarrow{i_1 i_2}, \overrightarrow{i_3 i_4}, \overrightarrow{i_5 i_6}, \overrightarrow{i_7 i_8}, \overrightarrow{i_9 i_{10}}, \overrightarrow{i_{11} i_{12}}$ and τ corresponds to the perfect matching $\overrightarrow{i_2 i_3}, \overrightarrow{i_4 i_5}, \overrightarrow{i_6 i_1}, \overrightarrow{i_8 i_9}, \overrightarrow{i_{10} i_7}, \overrightarrow{i_{11} i_{12}}$.

We have

$$(\operatorname{sgn} I)(\operatorname{sgn} J) = (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau) = \operatorname{sgn}\left(\tau\sigma^{-1}\right).$$

However, $\tau\sigma^{-1}$ is the permutation that is the product of k even cycles, so

$$\operatorname{sgn}\left(\tau\sigma^{-1}\right) = (-1)^k.$$

For the example on Fig. 4.5, we have $\tau\sigma^{-1} = (i_1 i_2 i_3 i_4 i_5 i_6)\,(i_7 i_8 i_9 i_{10})$ □

4.3.4 Theorem. *Let A be a skew-symmetric $n \times n$ matrix, where $n = 2m$ is an even integer. Then*

$$(\operatorname{Pf} A)^2 = \det A.$$

Proof. The result immediately follows from (4.3.1.3) and the fact that $\det K = 1$ for the matrix K defined by (4.3.1.4). It is instructive, however, to give a combinatorial proof along the lines of the proof of Theorem 4.1.4.

Let $\vec{G}$ be a complete directed graph with set $\{1, \ldots, n\}$ of vertices and edges $\vec{ij}$ for all pairs i, j including $i = j$. We introduce weights a_{ij} on the edges $\vec{ij}$ (in particular, loops $\vec{ii}$ have weight 0).

We write

$$\det A = \sum_{\vec{\Gamma}} (\operatorname{sgn} \vec{\Gamma}) \prod_{\vec{ij} \in \vec{\Gamma}} a_{ij}, \tag{4.3.4.1}$$

where the sum is taken over all directed cycle covers $\vec{\Gamma}$ of $\vec{G}$ and $\operatorname{sgn} \vec{\Gamma}$ is defined as the sign of the corresponding permutation, cf. Sect. 3.1.3. Note that $\operatorname{sgn} \vec{\Gamma}$ depends only on the cycle structure of $\vec{\Gamma}$, that is, on the number cycles of each length.

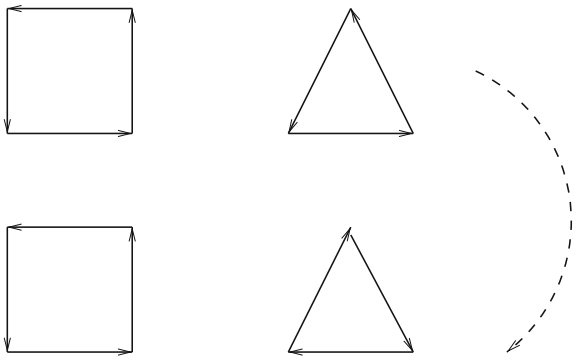


Fig. 4.6 Reversing the orientation of an odd cycle

Suppose that $\vec{\Gamma}$ contains a cycle of an odd length (an odd cycle). Since A is skew-symmetric, reversing the orientation of an odd cycle changes the sign of the corresponding term in (4.3.4.1), cf. Fig. 4.6.

Consequently, cycle covers $\vec{\Gamma}$ in (4.3.4.1) containing an odd cycle cancel each other out, and so we can write

$$\det A = \sum_{\vec{\Gamma} \text{ has no odd cycles}} (-1)^{c(\vec{\Gamma})} \prod_{\vec{ij} \in \vec{\Gamma}} a_{ij}, \tag{4.3.4.2}$$

where $c(\vec{\Gamma})$ is the number of cycles in $\vec{\Gamma}$.

Next, let G be the complete undirected graph with set $\{1, \dots, n\}$ of vertices and no loops and let $\widehat{G}$ be a directed graph obtained by orienting the edges of G arbitrarily, so that for every pair $i \neq j$ exactly one edge $\vec{ij}$ or $\vec{ji}$ is included in $\widehat{G}$. Then we can write (4.3.1.1) as

$$\operatorname{Pf} A = \sum_{I = \left\{ \overrightarrow{i_1 i_2}, \dots, \overrightarrow{i_{n-1} i_n} \right\}} (\operatorname{sgn} I) a_{i_1 i_2} \cdots a_{i_{n-1} i_n},$$

where the sum is taken over all perfect matchings I of $\widehat{G}$, cf. Sect. 4.3.2. Consequently,

$$(\operatorname{Pf} A)^2 = \sum_{\substack{I = \left\{ \overrightarrow{i_1 i_2}, \dots, \overrightarrow{i_{n-1} i_n} \right\} \\ J = \left\{ \overrightarrow{j_1 j_2}, \dots, \overrightarrow{j_{n-1} j_n} \right\}}} (\operatorname{sgn} I)(\operatorname{sgn} J) a_{i_1 i_2} \cdots a_{i_{n-1} i_n} a_{j_1 j_2} \cdots a_{j_{n-1} j_n},$$

where the sum is taken over all ordered pairs (I, J) (we allow $I = J$ and count such pairs once). The union of edges $\overrightarrow{i_1 i_2}$, …, $\overrightarrow{i_{n-1} i_n}$, $\overrightarrow{j_1 j_2}$, …, $\overrightarrow{j_{n-1} j_n}$ is a cycle cover Γ of $\widehat{G}$, where each cycle has an even length, cf. Fig. 4.4. Let $c_{>2}(\Gamma)$ be the number of cycles of Γ of length greater than 2 (hence $c_{>2}(\Gamma) = 2$ for the

cycle cover on Fig. 4.4). Then Γ can be represented as an ordered union $I \cup J$ of vertex-disjoint perfect matchings I and J in $2^{c_{>2}(\Gamma)}$ ways. By Lemma 4.3.3, the product $(\operatorname{sgn} I)(\operatorname{sgn} J)$ is independent on the representation, which allows us to define

$$\epsilon(\Gamma) = (\operatorname{sgn} I)(\operatorname{sgn} J)$$

for perfect matchings I and J whose union is the cycle cover Γ. Hence we can write

$$(\operatorname{Pf} A)^2 = \sum_{\Gamma \text{ has no odd cycles}} \epsilon(\Gamma) 2^{c_{>2}(\Gamma)} \prod_{\vec{ij} \in \Gamma} a_{ij}, \tag{4.3.4.3}$$

Furthermore, since A is skew-symmetric, the cycle cover obtained by reversing the orientation of a single edge in $\widehat{G}$ contributes the same monomial to (4.3.4.3). Moreover, since the cycle cover Γ of the undirected graph G can be oriented in $2^{c_{>2}(\Gamma)}$ ways, we can rewrite (4.3.4.3) as

$$(\operatorname{Pf} A)^2 = \sum_{\vec{\Gamma} \text{ has no odd cycles}} \epsilon(\vec{\Gamma})(-1)^{c_2(\vec{\Gamma})} \prod_{\vec{ij} \in \vec{\Gamma}} a_{ij}, \tag{4.3.4.4}$$

where the sum is taken over all oriented cycle covers $\vec{\Gamma}$ of the complete directed graph $\vec{G}$ by even cycles and $c_2(\vec{\Gamma})$ is the number of 2-cycles in $\vec{\Gamma}$. By Lemma 4.3.3, $\epsilon(\vec{\Gamma}) = (-1)^{c_{>2}(\Gamma)}$ and comparing (4.3.4.2) and (4.3.4.4), we complete the proof. □

4.3.5 Pfaffian orientation. In view of Theorem 4.3.4, formula (4.3.1.3) and the fact that the Pfaffian can be efficiently computed, the following question is of interest: Given a $2m \times 2m$ symmetric matrix $A = (a_{ij})$ with zero diagonal, is it possible to reverse the signs of some of the entries of A (that is, replace some a_{ij} by $-a_{ij}$) so that the resulting matrix B is skew-symmetric and $\operatorname{haf} A = \operatorname{Pf} B$?

Given such a matrix A, let us consider an undirected graph G_A with set $\{1, \ldots, n\}$ of vertices and edges $\{i, j\}$ provided $a_{ij} \neq 0$. We obtain a skew-symmetric matrix B if for every unordered pair $\{i, j\}$ we reverse the sign of exactly one entry among a_{ij} and a_{ji}. This procedure is encoded by making the graph G_A directed: for every edges $\{i, j\}$ of G_A we introduce the directed edge $\vec{ij}$ if the sign of a_{ij} is *not* reversed. We denote the resulting directed graph by $\vec{G}_B$. If $\operatorname{haf} A = \operatorname{Pf} B$, we say that $\vec{G}_B$ is the *Pfaffian orientation* of G_A.

Our next goal is to sketch a proof the famous result of Kasteleyn [Ka63], see also [TF61], that if G_A is a planar graph then it has a Pfaffian orientation, which can be constructed efficiently. We follow [LP09].

We call an even cycle C in G_A *relevant* if the graph obtained by deleting from G_A the vertices of C and all adjacent edges contains a perfect matching.

4.3.6 Lemma. *Let $\vec{G}_B$ be an orientation of G_A. Suppose that every relevant cycle C is oddly oriented. Then* $\operatorname{sgn} I = \operatorname{sgn} J$ *for any two perfect matchings in $\vec{G}_B$.*

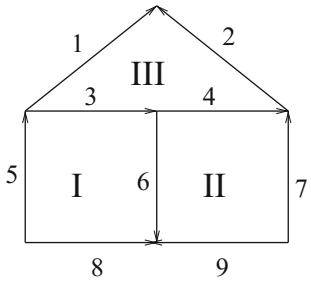


Fig. 4.7 A drawing of a planar graph

Proof. As follows from Lemma 4.3.3, $(\operatorname{sgn} I)(\operatorname{sgn} J) = 1$ for any two perfect matchings I and J. □

Let us consider a drawing of a directed planar graph $\vec{G}$ in the plane. Connected components of $\mathbb{R}^2 \setminus \vec{G}$ are called *faces* of $\vec{G}$. There is one unbounded face, and there can be no, one or several bounded faces. By choosing the orientation of the plane, we can talk about the edges of any bounded face oriented *clockwise* or *counterclockwise*, see Fig. 4.7.

For example, for the graph on Fig. 4.7, we have:

For the face I, the edges 3, 6 and 5 are oriented clockwise while the edge 8 is oriented counterclockwise;

For the face II, the edges 4 and 9 are oriented clockwise while the edges 6 and 7 is oriented counterclockwise;

For the face III, the edge 1 is oriented clockwise while the edges 3, 4 and 2 are oriented counterclockwise.

The set of edges $1, 2, 7, 9, 8, 5$ form a cycle C. With respect to that cycle, the edges 1, 9 and 5 are oriented clockwise while the edges 2, 7 and 8 are oriented counterclockwise.

Note that if the same edge belongs to two bounded faces then in one of the faces it is oriented clockwise and in the other counterclockwise.

Similarly, we define the orientation of the edges of any directed cycle drawn on the plane.

We will use the Euler formula relating the vertices, edges and faces of a planar graph G. To apply Euler's formula, we need the graph G to be *2-connected*, meaning that every two vertices of G can be connected by at least 2 vertex-disjoint (with the exception of the endpoints) paths in G, so that G has no "loose ends" and the embedding of G looks like the one on Fig. 4.8.

4.3.7 Lemma. *Let $\vec{G}$ be a drawing of a 2-connected directed graph, without loops or multiple edges, in the plane. Suppose that every bounded face has an odd number of edges oriented clockwise. Then every relevant cycle C is $\vec{G}$ oddly oriented.*

Proof. Since C is relevant, the graph obtained from $\vec{G}$ by deleting the vertices of C and all adjacent edges contains a perfect matching and, therefore, the number v of vertices of $\vec{G}$ lying inside the region bounded by C is even, so

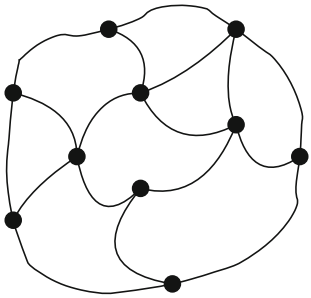


Fig. 4.8 A drawing of a 2-connected planar graph

$$v \equiv 0 \mod 2.$$

Let w be the number of vertices of C and hence also the number of edges of C. Let f be the number of faces lying inside C, let c_i be the number of clockwise oriented edges in the i-th face and let c be the number of clockwise edges in C. Since each c_i is odd, we have

$$\sum_{i=1}^{f} c_i \equiv f \mod 2.$$

Let e be the number of edges inside C. Then by the Euler's formula,

$$(v + w) - (e + w) + f = 1$$

and hence

$$e = v + f - 1$$

Since every interior edge is counted as clockwise for exactly one face, we have

$$\sum_{i=1}^{f} c_i = e + c$$

and hence

$$f \equiv e + c = c + v + f - 1 \mod 2.$$

It follows then that

$$c \equiv 1 \mod 2,$$

as required. □

Now it is clear how to construct a Pfaffian orientation of a planar graph: we build the graph edge by edge so that at most one new bounded face appears at each step. We orient the edge in such a way that the new face has an odd number of clockwise oriented edges, see Fig. 4.9.

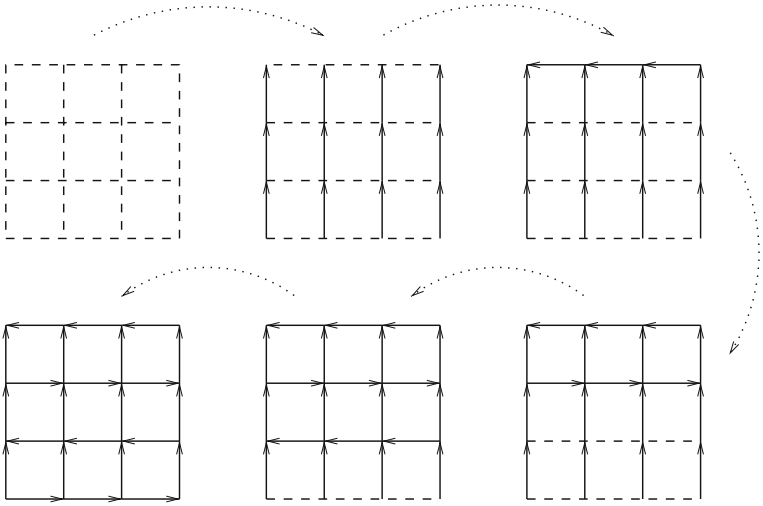


Fig. 4.9 Constructing a Pfaffian orientation of a graph

As follows by Lemmas 4.3.6 and 4.3.7, for any two perfect matchings I and J in the graph, we have $\operatorname{sgn} I = \operatorname{sgn} J$ and hence $\operatorname{haf} A = |\operatorname{Pf} B|$ for the skew-symmetric matrix B constructed from a given symmetric matrix A. If it so happens that $\operatorname{haf} A = -\operatorname{Pf} B$, we reverse the sign of the first row and column of B.

Galluccio and Loebl proved that if the genus of the graph G_A is g then $\operatorname{haf} A$ can be written as a sum of 4^g Pfaffians [GL99]. While no efficient algorithm for checking whether a given graph has a Pfaffian orientation appears to be known, in the case of a bipartite graph there is a polynomial time algorithm [R+99], see [Th06] for a survey.

4.3.8 Hafnians as expectations of random Pfaffians. Let $A = (a_{ij})$ be a non-negative real symmetric $n \times n$ matrix, where $n = 2m$ is even. For $1 \le i < j \le n$ let ξ_{ij} be real valued independent random variables such that

$$\mathbf{E}\,\xi_{ij} = 0 \quad \text{and} \quad \mathbf{var}\,\xi_{ij} = 1 \quad \text{for all} \quad 1 \le i < j \le n.$$

Let us define a skew-symmetric random matrix $B = (b_{ij})$ by

$$b_{ij} = \begin{cases} \xi_{ij}\sqrt{a_{ij}} & \text{if } i < j \\ -\xi_{ij}\sqrt{a_{ij}} & \text{if } i > j \\ 0 & \text{if } i = j. \end{cases}$$

It is not hard to see that

$$\operatorname{haf} A = \mathbf{E}\,(\operatorname{Pf} B)^2\,.$$

As in Sect. 3.9.1, the Markov inequality implies that the probability that $(\mathrm{Pf}\ B)^2$ overestimates haf A by a factor of $\lambda > 1$ does not exceed $1/\lambda$. In [Ba99] it is shown that if ξ_{ij} are independent standard Gaussian then with probability approaching 1 as n grows, we have

$$(\mathrm{Pf}\ B)^2 \ > \ c^n \,\mathrm{haf}\ A$$

for some absolute constant $c > 0$ (one can choose $c \approx 0.28$). As in Sect. 3.9, we can get a better constant $c \approx 0.56$ by switching to complex Gaussian ξ_{ij} and replacing $(\mathrm{Pf}\ B)^2$ by $|\mathrm{Pf}\ B|^2$, but unlike in the case of the permanent there does not seem to exist a viable quaternionic version of the estimator.

In [R+16], the authors identified a class of matrices A for which the approximation factor is subexponential in n.

4.4 Multidimensional Permanents

4.4.1 Permanents of tensors. Let $A = \left(a_{i_1\ldots i_d}\right)$ be a d-dimensional cubical $n \times \ldots \times n$ array (tensor) of complex numbers. We define the *d-dimensional permanent* of A by

$$\mathrm{PER}\ A = \sum_{\sigma_2,\ldots,\sigma_d \in S_n} \prod_{i=1}^{n} a_{i\sigma_2(i)\ldots\sigma_d(i)},$$

where the sum is taken over all $(d-1)$-tuples $(\sigma_2, \ldots, \sigma_d)$ of permutations sampled independently from the symmetric group S_n. In particular, if $d = 2$ then A is an $n \times n$ matrix and $\mathrm{PER}\ A = \mathrm{per}\ A$, cf. Sect. 3.1.1.

If $a_{i_1\ldots i_d} \in \{0, 1\}$ for all $1 \le i_1, \ldots, i_d \le n$, then PER A is naturally interpreted as the number of perfect matchings in the d-partite hypergraph H encoded by A: the vertices of H are split among d classes, where each class contains exactly n vertices, numbered $1, \ldots, n$ and the edges of H consist of the d-tuples $(i_1, \ldots, i_d)$ where $a_{i_1\ldots i_d} = 1$ and i_j denotes the i_j-th vertex from the j-th class. A *perfect matching* in H is a collection of edges containing each vertex exactly once.

For example, the perfect matching in a 3-partite hypergraph pictured on Fig. 4.10 corresponds to the pair of permutations (σ_2, σ_3), where

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

Hence for $d \ge 3$ it is an NP-hard problem to decide whether PER $A > 0$ for a given d-dimensional array A with 0-1 entries.

Given a d-dimensional array A, we call a (k, j)-th *slice* of A the set of all entries $a_{i_1\ldots i_d}$ where $i_k = j$. Hence if $d = 2$ a slice is a row ($k = 1$) or a column ($k = 2$) of the matrix A and for a general d, each entry of A is contained in exactly d slices and each slice consists of some n^{d-1} entries of A.

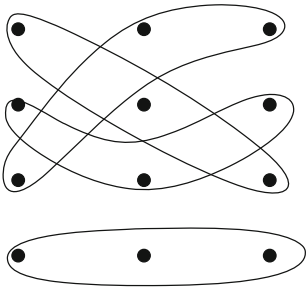


Fig. 4.10 A perfect matching in a 3-partite hypergraph

Similarly to (3.1.1.2), we obtain the "(1, 1)-slice expansion" of the permanent of a tensor:

$$\text{PER } A = \sum_{1 \le i_2, \ldots, i_d \le n} a_{1 i_2 \ldots i_d} \text{ PER } A_{i_2 \ldots i_d}, \tag{4.4.1.1}$$

where $A_{i_2 \ldots i_d}$ is the d-dimensional $(n-1) \times \ldots \times (n-1)$ array obtained from A by crossing out all slices containing $a_{1 i_2 \ldots i_d}$.

Some (but far from all) of the results and methods developed in Chap. 3 extend to multi-dimensional permanents. In particular, the permanents of tensors whose entries are close to 1 can be efficiently approximated, cf. Theorems 3.6.1 and 3.6.2.

4.4.2 Theorem. *For an integer $d \ge 2$ let us choose*

$$\delta_d = \sin \frac{\alpha}{2} \cos \frac{(d-1)\alpha}{2}$$

for some $\alpha = \alpha_d > 0$ such that

$$(d-1)\alpha < \frac{2\pi}{3}.$$

Hence $0 < \delta_d < 1$ and we can choose $\delta_2 = 0.5$, $\delta_3 = \sqrt{6}/9 \approx 0.272$, $\delta_4 \approx 0.1845$ and $\delta_d = \Omega(1/d)$.

(1) For any d-dimensional cubical array $Z = \left(z_{i_1 \ldots i_d}\right)$ of complex numbers satisfying

$$\left|1 - z_{i_1 \ldots i_d}\right| \le \delta_d \quad \text{for all} \quad 1 \le i_1, \ldots, i_d \le n$$

we have

$$\text{PER } Z \neq 0.$$

(2) For any $0 < \delta < \delta_d$ there is $\gamma = \gamma(\delta_d/\delta) > 0$ and for any $\epsilon > 0$ and integer $n \ge 1$ there is a polynomial $p = p_{d,n,\epsilon,\delta}$ in the entries of a d-dimensional $n \times \ldots \times n$ array A such that

$$\deg p \ \leq \ \gamma(\ln n - \ln \epsilon)$$

and

$$|\ln \text{PER } A - p(A)| \ \leq \ \epsilon$$

provided $A = \left(a_{i_1 \ldots i_d}\right)$ *is a cubical* d*-dimensional* $n \times \ldots \times n$ *array of complex numbers satisfying*

$$\left|1 - a_{i_1 \ldots i_d}\right| < \delta \quad \textit{for all} \quad 1 \leq i_1, \ldots, i_d \leq n.$$

Proof. The proof is similar to those of Sect. 3.6. In Part (1), let $\alpha = \alpha_d$ be a real number such that $0 < (d-1)\alpha < 2\pi/3$ and

$$\delta_d = \sin \frac{\alpha}{2} \cos \frac{(d-1)\alpha}{2}.$$

We prove by induction on n that if $A = \left(a_{i_1 \ldots i_d}\right)$ and $B = \left(b_{i_1 \ldots i_d}\right)$ are two $n \times \ldots \times n$ arrays of complex numbers satisfying

$$\left|1 - a_{i_1 \ldots i_d}\right| \ \leq \ \delta_d \quad \text{and} \quad \left|1 - b_{i_1 \ldots i_d}\right| \ \leq \ \delta_d$$

for all $1 \leq i_1, \ldots, i_d \leq n$ and such that A and B differ in at most one slice, then PER $A \neq 0$, PER $B \neq 0$ and the angle between two non-zero complex numbers PER A and PER B does not exceed α.

For $n = 1$, then clearly PER $A \neq 0$ and PER $B \neq 0$ and the angle between the two numbers does not exceed $2 \arcsin \delta_d < \alpha$. Assuming that $n > 1$, without loss of generality, we assume that B is obtained from A by replacing $a_{1 i_2 \ldots, i_d}$ by $b_{1 i_2 \ldots i_d}$ for all $1 \leq i_2, \ldots, i_d \leq n$. By (4.4.1.1), we have

$$\text{PER } A = \sum_{1 \leq i_2, \ldots, i_d \leq n} a_{1 i_2 \ldots i_d} \text{ PER } A_{i_2 \ldots i_d} \quad \text{and}$$

$$\text{PER } B = \sum_{1 \leq i_2, \ldots, i_d \leq n} b_{1 i_2 \ldots i_d} \text{ PER } A_{i_2 \ldots i_d},$$

where $A_{i_2 \ldots i_d}$ is the $(n-1) \times \ldots \times (n-1)$ array obtained from A by crossing out all d slices containing $a_{1 i_2 \ldots i_d}$. Next, we observe that any two arrays $A_{i_2 \ldots i_d}$ and $A_{j_2 \ldots j_d}$, up to a permutation of slices, differ in at most $(d-1)$ slices. By the induction hypothesis we have PER $A_{i_2 \ldots i_d} \neq 0$ for all $1 \leq i_2, \ldots, i_d \leq n$ and the angle between any two non-zero complex numbers PER $A_{i_2 \ldots i_d}$ and PER $A_{j_2 \ldots j_d}$ does not exceed $(d-1)\alpha$. Applying Lemma 3.6.4, we conclude that PER $A \neq 0$, PER $B \neq 0$ and the angle between PER A and PER B does not exceed

$$2 \arcsin \frac{\delta_d}{\cos \frac{(d-1)\alpha}{2}} = \alpha,$$

which completes the proof of Part (1).

To prove Part (2), let $J = J_{d,n}$ be the $n \times \ldots \times n$ tensor filled with 1s. We define a polynomial

$$g(z) = \text{PER}\,(J + z(A - J))$$

of degree at most n, so that

$$g(0) = \text{PER}\ J = (n!)^{d-1} \quad \text{and} \quad g(1) = \text{PER}\ A.$$

Moreover, for $\beta = \delta_d/\delta > 1$ we have $g(z) \neq 0$ whenever $|z| \leq \beta$. We choose a branch of $f(z) = \ln g(z)$ for $|z| \leq 1$ such that $f(0)$ is real and use Lemma 2.2.1 to claim that for some $k \leq \gamma(\ln n - \ln \epsilon)$ the Taylor polynomial

$$p_k(z) = f(0) + \sum_{m=1}^{k} \frac{f^{(m)}(0)}{m!} z^m$$

approximates $f(z)$ for $|z| \leq 1$ within an additive error ϵ. We need to show that $p_k(1)$ is a polynomial of degree at most k in the entries of A. To finish the proof as in Sect. 3.6.7, it suffices to show that $g^{(m)}(0)$ is a polynomial in the entries of A of degree at most m. Indeed,

$$\begin{aligned} g^{(m)}(0) &= \frac{d^m}{dz^m} \sum_{\sigma_2,\ldots,\sigma_d \in S_n} \prod_{i=1}^{n} \left(1 + z\left(a_{i\sigma_2(i)\ldots\sigma_d(i)} - 1\right)\right)\Big|_{z=0} \\ &= \sum_{\sigma_2,\ldots,\sigma_d \in S_n} \sum_{(i_1,\ldots,i_m)} \left(a_{i_1\sigma_2(i_1)\ldots\sigma_d(i_1)} - 1\right) \cdots \left(a_{i_m\sigma_2(i_m)\ldots\sigma_d(i_m)} - 1\right), \end{aligned}$$

where the last sum is taken over all ordered m-tuples of distinct indices $(i_1, \ldots, i_m)$. Therefore,

$$\begin{aligned} g^{(m)}(0) = ((n-m)!)^{d-1} \\ \times \sum_{\substack{(i_{11},\ldots,i_{1m}) \\ (i_{21},\ldots,i_{2m}) \\ \ldots\ldots\ldots \\ (i_{d1},\ldots,i_{dm})}} \left(a_{i_{11}i_{21}\ldots i_{d1}} - 1\right)\left(a_{i_{12}i_{22}\ldots i_{d2}} - 1\right) \cdots \left(a_{i_{1m}i_{2m}\ldots i_{dm}} - 1\right), \end{aligned}$$

where the sum is taken over ordered d-tuples of ordered m-tuples $\left(i_{j1}, \ldots, i_{jm}\right)$ for $1 \leq j \leq d$ of distinct indices i_{jk}. □

For fixed d and δ, the polynomial p can be computed in $n^{O(\ln n - \ln \epsilon)}$ time. Later, in Theorem 5.5.3, we prove that PER $A \neq 0$ if the ℓ^1 distance of each slice of a d-dimensional $n \times \ldots \times n$ complex cubical array A to the array of 1 s does not exceed $\gamma_d n^{d-1}$, where $\gamma_d = (\alpha(d-1))^{d-1}/d^d$ and $\alpha \approx 0.278$ is an absolute constant.

If the entries of the tensor are real positive, we obtain better bounds, although for $d > 2$ the improvement is not as substantial as in the case of permanents, see Sect. 3.7.

4.4.3 Theorem. *For an integer $d \geq 2$, let*

$$\delta_d = \tan \frac{\pi}{4(d-1)}$$

so that $\delta_2 = 1$, $\delta_3 = \sqrt{2} - 1 \approx 0.41$, $\delta_4 = 2 - \sqrt{3} \approx 0.27$, etc.

(1) Let us fix real δ and τ where

$$0 \leq \delta < \delta_d \quad \text{and} \quad \tau = (1-\delta) \sin \left(\frac{\pi}{4(d-1)} - \arctan \delta \right) > 0$$

Let $Z = \left(z_{i_1 \ldots i_d}\right)$ be a d-dimensional $n \times \cdots \times n$ array of complex numbers such that for all $1 \leq i_1, \ldots, i_d \leq n$ we have

$$\left|1 - \Re\, z_{i_1 \ldots i_d}\right| \;\leq\; \delta \quad \text{and} \quad \left|\Im\, z_{i_1 \ldots i_d}\right| \;\leq\; \tau.$$

Then

$$\text{PER } Z \neq 0.$$

(2) For any integer $d \geq 2$ and any $0 \leq \delta < \delta_d$ there is a constant $\gamma = \gamma(\delta_d/\delta) > 0$ and for any positive integer n and real $0 < \epsilon < 1$ there is a polynomial $p = p_{n,d,\delta,\epsilon}$ of $\deg p \leq \gamma\,(\ln n - \ln \epsilon)$ in the entries of a d-dimensional $n \times \cdots \times n$ array such that

$$|\ln \text{PER } A - p(A)| \;\leq\; \epsilon$$

for any d-dimensional $n \times \cdots \times n$ array $A = \left(a_{i_1 \ldots i_d}\right)$ of real numbers satisfying

$$\left|1 - a_{i_1 \ldots i_d}\right| \;\leq\; \delta \quad \text{for all} \quad 1 \leq i_1, \ldots, i_d \leq n.$$

Proof. The proof is similar to those of Sect. 3.7. Let $\mathcal{U}_n = \mathcal{U}_n(d, \delta, \tau)$ be the set of all d-dimensional $n \times \ldots \times n$ complex tensors $Z = \left(z_{i_1 \ldots i_d}\right)$ that satisfy the conditions of Part (1). We prove by induction on n that for any two tensors $A, B \in \mathcal{U}_n$ that differ in at most one slice, we have PER $A \neq 0$, PER $B \neq 0$ and the angle between the complex numbers does not exceed $\dfrac{\pi}{2(d-1)}$.

If $n = 1$ then clearly PER $A \neq 0$, PER $B \neq 0$ and the angle between PER A and PER B does not exceed

$$2\arctan\frac{\tau}{1-\delta} \;\le\; \frac{\pi}{2(d-1)} - 2\arctan\delta \;<\; \frac{\pi}{2(d-1)}.$$

Suppose that $n > 1$. Without loss of generality, we assume that B is obtained by A by replacing $a_{1i_2\ldots i_d}$ by $b_{1i_2\ldots i_d}$ for all $1 \le i_2, \ldots, i_d \le n$. We have

$$\begin{aligned}\text{PER } A &= \sum_{1\le i_2,\ldots,i_d\le n} a_{1i_2\ldots i_d}\,\text{PER } A_{i_2\ldots i_d} \quad \text{and}\\ \text{PER } B &= \sum_{1\le i_2,\ldots,i_d\le n} b_{1i_2\ldots i_d}\,\text{PER } A_{i_2\ldots i_d},\end{aligned}$$

where $A_{i_2\ldots i_d}$ is the $(n-1)\times\ldots\times(n-1)$ array obtained from A by crossing out all d slices containing $a_{1i_2\ldots i_d}$. Next, we observe that any two arrays $A_{i_2\ldots i_d}$ and $A_{j_2\ldots j_d}$, up to a permutation of slices, differ in at most $(d-1)$ slices. By the induction hypothesis, we have PER $A_{i_2\ldots i_d} \ne 0$ for all $1 \le i_2, \ldots, i_d \le n$ and that the angle between any two non-zero complex numbers PER $A_{i_2\ldots i_d}$ and PER $A_{j_2\ldots j_d}$ does not exceed $\pi/2$. Applying Part (3) of Lemma 3.7.3, we conclude that PER $A \ne 0$, PER $B \ne 0$ and that the angle between PER A and PER B does not exceed

$$2\arctan\delta + 2\arcsin\frac{\tau}{1-\delta} = \frac{\pi}{2(d-1)}.$$

To prove Part (2), Let $J = J_{n,d}$ be the $n\times\cdots\times n$ tensor filled with 1 s and let us define a univariate polynomial

$$r(z) = \text{PER}\,(J + z(A-J)).$$

Suppose that

$$-\alpha \;\le\; \Re\, z \;\le\; 1+\alpha \quad\text{and}\quad |\Im\, z| \;\le\; \rho \tag{4.4.3.1}$$

for some $\alpha > 0$ and $\rho > 0$. Then

$$\begin{aligned}1-(1+\alpha)\delta \;\le\; \Re\left(1+z\left(a_{i_1\ldots i_d}-1\right)\right) \;\le 1+(1+\alpha)\delta \quad\text{and}\\ \left|\Im\left(1+z\left(a_{i_1\ldots i_d}-1\right)\right)\right| \;\le \rho\delta.\end{aligned}$$

Let us choose a sufficiently small $\alpha = \alpha(\delta) > 0$ so that $\delta' = (1+\alpha)\delta < \delta_d$ and let

$$\rho = \frac{1-\delta'}{\delta}\sin\left(\frac{\pi}{4(d-1)} - \arctan\delta'\right).$$

Then by Part (1) we have $r(z) \ne 0$ for all z satisfying (4.4.3.1). Let $\phi(z) = \phi_{\delta_d/\delta}(z)$ be a univariate polynomial constructed in Lemma 2.2.3, such that

$$\phi(0) = 0, \quad \phi(1) = 1$$

and

$$-\alpha \;\le\; \Re\,\phi(z) \;\le\; 1+\alpha \quad \text{and} \quad |\Im\,\phi(z)| \;\le\; \rho$$

provided

$$|z| \;\le\; \beta \quad \text{for some} \quad \beta = \beta(\delta_d/\delta) \;>\; 1.$$

We define the composition

$$g(z) = r(\phi(z)).$$

Then $g(z)$ is a univariate polynomial and $\deg g = (\deg r)(\deg \phi) = O(n)$, where the implicit constant in the "O" notation depends only on δ_d/δ. In addition,

$$g(0) = r(0) = \text{PER } J = (n!)^{d-1} \quad \text{and} \quad g(1) = r(1) = \text{PER } A$$

and

$$g(z) \neq 0 \quad \text{provided} \quad |z| \;\le\; \beta.$$

The proof is finished as in Sect. 3.7.5. We choose a branch of $f(z) = \ln g(z)$ in the disc $|z| \le 1$ so that

$$f(0) = (d-1)\ln n! \quad \text{and} \quad f(1) = \ln \text{PER } A.$$

Let $p_m(z)$ be the Taylor polynomial of $f(z)$ of degree m computed at $z = 0$. By Lemma 2.2.1, we can choose $m = O(\ln n - \ln \epsilon)$, where the implicit constant in the "O" notation depends on δ_d/δ, so that $p_m(1)$ approximates $f(1)$ within an additive error of ϵ. It remains to show that the k-th derivative $f^{(k)}(0)$ is a polynomial of degree k in the entries of the tensor A. From Sect. 2.2.2, it suffices to show that $g^{(k)}(0)$ is a polynomial of degree k in the entries of the tensor A. We showed in the proof of Theorem 4.4.2 that $r^{(k)}(0)$ is a polynomial in the entries of A of degree k and we compute the expansion of the composition $g(z) = r(\phi(z))$ as in Sect. 3.7.5. □

For fixed d and δ, the polynomial p in Part (2) of Theorem 4.4.3 can be computed in $n^{O(\ln n - \ln \epsilon)}$ time.

Figure 4.11 pictures regions for the entries of a tensor allowable by Theorem 4.4.2 (disc) and allowable by Theorem 4.4.3 (rectangle).

4.4.4 Scaling. Similarly to the scaling of matrices (see Sect. 3.5), one can define the scaling of tensors. We say that the d-dimensional $n \times \ldots \times n$ tensor $A = \left(a_{i_1 \ldots i_d}\right)$ is obtained from the d-dimensional $n \times \ldots \times n$ tensor $B = \left(b_{i_1 \ldots i_d}\right)$ by *scaling* if there exist $\lambda_{kj} > 0$ for $k = 1, \ldots, d$ and $j = 1, \ldots, n$ such that

$$a_{i_1 \ldots i_d} = \lambda_{1 i_1} \cdots \lambda_{d i_d} b_{i_1 \ldots i_d} \quad \text{for all} \quad 1 \le i_1, \ldots, i_d \le n.$$

Clearly, in this case,

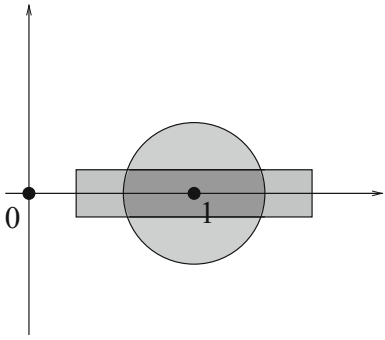


Fig. 4.11 A neighborhood for the entries of A (disc) where PER A is approximated by Theorem 4.4.2 and a neighborhood for the entries of A (rectangle) where PER A is approximated by Theorem 4.4.3

$$\text{PER } A = \left(\prod_{\substack{1 \le k \le d \\ 1 \le j \le n}} \lambda_{kj} \right) \text{PER } B. \tag{4.4.4.1}$$

We say that B is *d-stochastic* if the entries of B are non-negative:

$$b_{i_1 \dots i_d} \ \ge \ 0 \quad \text{for all} \quad 1 \le i_1, \dots, i_d \le n$$

and the sum of entries in every slice is 1:

$$\sum_{1 \le i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_d \le n} b_{i_1 \dots i_{k-1} i i_{k+1} \dots i_d} = 1$$
$$\text{for all} \quad k = 1, \dots, d \quad \text{and all} \quad i = 1, \dots, n. \tag{4.4.4.2}$$

4.4.5 Theorem. *Any d-dimensional cubical tensor* $A = \left(a_{i_1 \dots i_d}\right)$ *with real positive entries*

$$a_{i_1 \dots, i_d} > 0 \ \ \textit{for all} \ \ 1 \le i_1, \dots, i_d \le n$$

can be obtained by scaling from a unique d-stochastic tensor B. The tensor B can be found as a necessarily unique minimum of the convex function

$$f(X) = \sum_{1 \le i_1, \dots, i_d \le n} x_{i_1 \dots i_d} \ln \frac{x_{i_1 \dots i_d}}{a_{i_1 \dots i_d}}$$

on the convex polytope of d-stochastic $n \times \dots \times n$ *tensors X. Thus we have*

$$b_{i_1 \dots i_d} = \lambda_{1 i_1} \cdots \lambda_{d i_d} a_{i_1 \dots i_d} \ \ \textit{for all} \ \ 1 \le i_1, \dots, i_d \le n$$

and some $\lambda_{kj} > 0$ *for* $k = 1, \dots, d$ *and* $j = 1, \dots, n$. *The numbers* λ_{kj} *are unique up to a rescaling*

$$\lambda_{kj} \longrightarrow \tau_k \lambda_{kj} \quad \textit{for all} \quad k, j,$$

for some $\tau_1, \ldots, \tau_d > 0$ *such that* $\tau_1 \cdots \tau_d = 1$.

The proof is very similar to the proof of Theorem 3.5.2, see [BS11] and also [Fr11] for extensions in the case of non-negative tensors A. We note that

$$f(B) = \sum_{1 \le i_1, \ldots, i_d \le n} b_{i_1 \ldots i_d} \ln \frac{b_{i_1 \ldots i_d}}{a_{i_1 \ldots i_d}} = \sum_{1 \le i_1, \ldots, i_d \le n} b_{i_1 \ldots i_d} \left(\sum_{k=1}^{d} \ln \lambda_{k i_k} \right)$$
$$= \sum_{k=1}^{d} \sum_{j=1}^{n} \ln \lambda_{kj} \left(\sum_{\substack{1 \le i_1, \ldots, i_d \le n \\ i_k = j}} b_{i_1 \ldots i_d} \right) = \sum_{k=1}^{d} \sum_{j=1}^{n} \ln \lambda_{kj}.$$

For $d \ge 3$, it is relatively easy to construct an example of a d-stochastic tensor B such that PER $B = 0$, see [BS11]. The situation with the d-dimensional permanent is somewhat similar to that with the hafnian, cf. Sect. 4.2: while there is no van der Waerden-type lower bound, there is concentration of the permanents of well-conditioned d-stochastic tensors.

4.4.6 Definition. Let $A = \left(a_{i_1 \ldots i_d}\right)$ be a d-dimensional tensor with positive entries. For $\alpha \ge 1$, we say that A is *α-conditioned* if

$$a_{i_1 \ldots i_k \ldots i_d} \le \alpha a_{i_1 \ldots i'_k \ldots i_d} \quad \text{for all} \quad 1 \le i_1, \ldots, i_k, i'_k, \ldots, i_d \le n \quad \text{and all} \quad k = 1, \ldots, d. \tag{4.4.6.1}$$

In words: a tensor with positive entries is α-conditioned if the ratio of any two entries which differ in one index does not exceed α.

4.4.7 Lemma. *Let A be an α-conditioned d-dimensional cubical tensor and let B be a d-stochastic tensor obtained from A by scaling. Then B is α^2-conditioned.*

The proof is very similar to that of Lemma 3.5.6, see also [BS11] for details.

Our next goal is to prove the concentration of d-dimensional permanents of well-conditioned d-stochastic tensors.

4.4.8 Theorem. *For any real $\alpha \ge 1$ and any integer $d > 1$ there exists $\gamma = \gamma(d, \alpha) > 0$ such that if A is an α-conditioned d-stochastic $n \times \ldots \times n$ array then*

$$n^{-\gamma} e^{-n(d-1)} \le \text{PER } A \le n^{\gamma} e^{-n(d-1)}.$$

The proof follows the same scheme as the proof of Theorem 4.2.2 for hafnians, see also [BS11].

First, we need the dual description of the scaling factors λ_{kj} in Theorem 4.4.5, cf. Theorem 4.2.4.

4.4.9 Lemma. *Let $A = \left(a_{i_1 \ldots i_d}\right)$ be a d-dimensional $n \times \ldots \times n$ tensor with positive entries and let $\lambda_{kj} > 0 : 1 \leq k \leq d$, $1 \leq j \leq n$ be real numbers such that the tensor $B = \left(b_{i_1 \ldots i_d}\right)$ where*

$$b_{i_1 \ldots i_d} = \lambda_{1 i_1} \cdots \lambda_{d i_d} a_{i_1 \ldots i_d} \quad \text{for all} \quad 1 \leq i_1, \ldots, i_d \leq n$$

is d-stochastic. Then the point

$$\xi_{kj} = \ln \lambda_{kj}$$

is a maximum point of the linear function $\ell : \mathbb{R}^{d \times n} \longrightarrow \mathbb{R}$,

$$\ell(x) = \sum_{k=1}^{d} \sum_{j=1}^{n} x_{kj} \quad \text{for} \quad x = \left(x_{kj}\right)$$

on the convex set $\mathcal{C} = \mathcal{C}(A) \subset \mathbb{R}^{d \times n}$ defined by the inequality

$$\mathcal{C} = \left\{ x = \left(x_{kj}\right) : \sum_{1 \leq i_1, \ldots, i_d \leq n} a_{i_1 \ldots i_d} \exp \left\{ \sum_{k=1}^{d} \sum_{j=1}^{n} x_{kj} \right\} \leq n \right\}.$$

The proof is similar to that of Theorem 4.2.4.

Next, we show that if a d-dimensional tensor which is close to d-stochastic is scaled to d-stochastic, then the product of the scaling factors is close to 1.

4.4.10 Lemma. *Let $A = \left(a_{i_1 \ldots i_d}\right)$ be an α-conditioned d-dimensional $n \times \ldots \times n$ tensor such that the sum of entries of A in the (k, j)-th slice is $1 - \delta_{kj}$, where*

$$\left|\delta_{kj}\right| \leq \frac{\beta}{n} \quad \text{for} \quad k = 1, \ldots, d \quad \text{and} \quad j = 1, \ldots, n$$

and some

$$0 \leq \beta \leq \frac{n}{\alpha^{d-1} d}.$$

Suppose further that the sum of the entries of A is n. Let $B = \left(b_{i_1 \ldots i_d}\right)$ be a d-stochastic tensor obtained from A by scaling, so that

$$b_{i_1 \ldots i_d} = \lambda_{1 i_1} \cdots \lambda_{d i_d} a_{i_1 \ldots i_d} \quad \text{for all} \quad 1 \leq i_1, \ldots, i_d \leq n$$

and some $\lambda_{kj} > 0$. Then

$$0 \leq \sum_{k=1}^{d} \sum_{j=1}^{n} \ln \lambda_{kj} \leq \frac{\alpha^{d-1} \beta^2 d^2}{n}.$$

Proof. Since the point $x_{kj} = 0$ belongs to the convex set $\mathcal{C}$ of Lemma 4.4.9, we conclude that

$$\sum_{k=1}^{d}\sum_{j=1}^{n} \ln \lambda_{kj} \ \geq \ 0.$$

Since the sum of entries of A is n, we have

$$\sum_{j=1}^{n} \delta_{kj} = 0 \quad \text{for} \quad k = 1, \ldots, d. \tag{4.4.10.1}$$

Let us define a tensor $X = \left(x_{i_1 \ldots i_d}\right)$ by

$$x_{i_1 \ldots i_d} = a_{i_1 \ldots i_d} + w_{i_1 \ldots i_d} \quad \text{where} \quad w_{i_1 \ldots i_d} = \frac{1}{n^{d-1}} \sum_{k=1}^{d} \delta_{k i_k}.$$

It follows by (4.4.10.1) that the sum of entries of X in every slice is 1. Since A is α-conditioned, we have

$$a_{i_1 \ldots i_d} \ \geq \ \frac{1}{(\alpha n)^{d-1}} \quad \text{for all} \quad i_1, \ldots, i_d \tag{4.4.10.2}$$

and hence X is d-stochastic. From Theorem 4.4.5,

$$\begin{aligned}
\sum_{k=1}^{d}\sum_{j=1}^{n} \ln \lambda_{kj} \ &\leq \sum_{1 \leq i_1, \ldots, i_d \leq n} x_{i_1 \ldots i_d} \ln \frac{x_{i_1 \ldots i_d}}{a_{i_1 \ldots i_d}} \\
&= \sum_{1 \leq i_1, \ldots, i_d \leq n} \left(a_{i_1 \ldots i_d} + w_{i_1 \ldots i_d}\right) \ln \left(1 + \frac{w_{i_1 \ldots i_d}}{a_{i_1 \ldots i_d}}\right) \\
&\leq \sum_{1 \leq i_1, \ldots, i_d \leq n} \left(a_{i_1 \ldots i_d} + w_{i_1 \ldots i_d}\right) \frac{w_{i_1 \ldots i_d}}{a_{i_1 \ldots i_d}} \\
&= \sum_{1 \leq i_1, \ldots, i_d \leq n} \frac{w^2_{i_1 \ldots i_d}}{a_{i_1 \ldots i_d}}.
\end{aligned}$$

Since

$$\left|w_{i_1 \ldots i_d}\right| \ \leq \ \frac{\beta d}{n^d},$$

by (4.4.10.2) we conclude that

$$\sum_{k=1}^{d}\sum_{j=1}^{n}\ln\lambda_{kj} \;\le\; \sum_{1\le i_1,\ldots,i_d\le n}\frac{w^2_{i_1\ldots i_d}}{a_{i_1\ldots i_d}} \;\le\; n^d\frac{\beta^2d^2}{n^{2d}}(\alpha n)^{d-1} = \frac{\alpha^{d-1}\beta^2d^2}{n}.$$

□

Now we are ready to prove Theorem 4.4.8.

4.4.11 Proof of Theorem 4.4.8**.** All implied constants in the "O" notation below depend on α and d only.

For subsets $I_1, I_2, \ldots, I_d \subset \{1,\ldots,n\}$ such that $|I_1| = \ldots = |I_d|$, we denote by $A(I_1,\ldots,I_d)$ the d-dimensional tensor consisting of the entries $a_{i_1\ldots i_d}$ where $i_k \in I_k$ for $k = 1,\ldots,d$. Let $B(I_1,\ldots,I_d)$ be the d-stochastic tensor obtained from $A(I_1,\ldots,I_d)$ by scaling. We prove by induction on $m = |I_1| = \ldots = |I_d|$ that

$$\text{PER } B(I_1,\ldots,I_d) = \exp\left\{-m(d-1) + O\left(\sum_{j=1}^{m}\frac{1}{j}\right)\right\} \tag{4.4.11.1}$$

Substituting $m = n$, we get the desired result.

Let $I_1,\ldots,I_d \subset \{1,\ldots,n\}$ be subsets such that $|I_1| = \ldots = |I_d| = m$ and let us choose $i_1 \in I_1$. To simplify the notation, we denote $B(I_1,\ldots,I_d)$ just by B and also assume that $i_1 = 1$. We use the $(1,1)$-slice expansion (4.4.1.1):

$$\text{PER } B = \sum_{i_2\in I_2,\ldots,i_d\in I_d} b_{1i_2\ldots i_d}\,\text{PER } B_{i_2\ldots i_d}, \tag{4.4.11.2}$$

where $B_{i_2\ldots i_d}$ is the tensor obtained from B by crossing out all slices containing $b_{1i_2\ldots i_d}$. Note that (4.4.11.2) represents PER B as a convex combination of PER $B_{i_2\ldots i_d}$.

By Lemma 4.4.7, the tensor B is α^2-conditioned. Since B is d-stochastic, the entries of B do not exceed $\alpha^{2(d-1)}/m^{d-1}$. Let $\sigma_{i_2\ldots i_d}$ be the sum of the entries of $B_{i_2\ldots i_d}$. Hence

$$\sigma_{i_2\ldots i_d} = m - d + O\left(\frac{1}{m}\right) \tag{4.4.11.3}$$

(we obtain a lower bound when we subtract from the total sum of the entries of B the sums over d slices and we obtain an upper bound if we add back the sums over all pairwise intersections of slices).

We scale $B_{i_2\ldots i_d}$ to the total sum of entries $m-1$, so we define

$$\widehat{B}_{i_2\ldots i_d} = \frac{m-1}{\sigma_{i_2\ldots i_d}}B_{i_2\ldots i_d}.$$

Then

$$\text{PER } B_{i_2\ldots i_d} = \left(\frac{\sigma_{i_2\ldots i_d}}{m-1}\right)^{m-1} \text{PER } \widehat{B}_{i_2\ldots i_d}$$

and by (4.4.11.3), we conclude that

$$\text{PER } B_{i_2\ldots i_d} = \exp\left\{-(d-1) + O\left(\frac{1}{m}\right)\right\} \text{PER } \widehat{B}_{i_2\ldots i_d}. \tag{4.4.11.4}$$

To estimate PER $\widehat{B}_{i_2\ldots i_d}$ we use Lemma 4.4.10. Let us scale $\widehat{B}_{i_2\ldots i_d}$ to a d-stochastic tensor. The resulting d-stochastic tensor is the same tensor we obtain from $A(I \setminus \{i_1\}, \ldots, I_d \setminus \{i_d\})$ by scaling, that is, the tensor $B(I_1 \setminus \{i_1\}, \ldots, I_d \setminus \{i_d\})$. Since the tensor B is d stochastic and the entries of B do not exceed $\alpha^{2(d-1)}/m^{d-1}$, we conclude that the sum of entries in every slice of $B_{i_2\ldots i_d}$ is at most 1 and at least $1 - \alpha^{2(d-1)}/m$. Consequently, the absolute value of the difference of the sum of entries in every slice $\widehat{B}_{i_2\ldots i_d}$ and 1 is $O(1/m)$. Applying Lemma 4.4.10, we conclude that as long as $m > \gamma_1(\alpha, d)$ for some constant γ_1 depending on α and d only, we have

$$\text{PER } \widehat{B}_{i_2\ldots i_d} = \exp\left\{O\left(\frac{1}{m}\right)\right\} \text{PER } B(I_1 \setminus \{i_1\}, \ldots, I_d \setminus \{i_d\}). \tag{4.4.11.5}$$

We use a trivial estimate

$$\text{PER } B = e^{O(1)} \quad \text{provided} \quad m \le \gamma_1(\alpha, d). \tag{4.4.11.6}$$

Applying the induction hypothesis to PER $B(I_1 \setminus \{i_1\}, \ldots, I_d \setminus \{i_d\})$ and combining (4.4.11.6), (4.4.11.5) and (4.4.11.2), we complete the proof of (4.4.11.1). □

4.4.12 Algorithmic applications. It follows from Theorem 4.4.5, Lemma 4.4.7 and Theorem 4.4.8 that for any $\alpha \ge 1$, fixed in advance, the permanent of a d-dimensional α-conditioned $n \times \ldots \times n$ tensor can be efficiently (in polynomial time) approximated within a polynomial in n factor of n^γ for some $\gamma = \gamma(\alpha, d)$. As is argued in [BS11], this allows us to distinguish d-partite hypergraphs that are far from having a perfect matchings from d-partite hypergraphs that have sufficiently many perfect matchings even when "sufficiently many" means that the probability to hit a perfect matching at random is exponentially small.

Let $V = V_1 \cup \ldots \cup V_d$ be the set of vertices of a d-partite hypergraph H, where $|V_1| = \ldots = |V_d| = n$ and for every edge S of H we have $|S \cap V_1| = \ldots = |S \cap V_d| = 1$. We identify each "part" V_i with a copy of the set $\{1, \ldots, n\}$, fix an $0 < \epsilon < 1$ and construct a d-dimensional $n \times \ldots \times n$ tensor $A = \left(a_{i_1\ldots i_d}\right)$ by

$$a_{i_1\ldots i_d} = \begin{cases} 1 & \text{if } (i_1, \ldots, i_d) \text{ is an edge of } H \\ \epsilon & \text{otherwise.} \end{cases}$$

Then H is $1/\epsilon$-conditioned, and applying Theorem 4.4.5, Lemma 4.4.7 and Theorem 4.4.8 we can estimate in polynomial time PER A within a multiplicative factor of $n^{\gamma(\epsilon,d)}$. Now, if every matching in H consists of at most $(1-\delta)n$ edges for some $\delta > 0$, we have PER $A \leq \epsilon^{\delta n}(n!)^{d-1}$. On the other hand, if H has at least $\beta^n(n!)^{d-1}$ perfect matchings for some $0 < \beta \leq 1$, we have PER $A \geq \beta^n(n!)^{d-1}$. As long as $\epsilon^{\delta} < \beta$, by computing PER A within a factor of $n^{\gamma(\epsilon,d)}$, we can distinguish these two cases.

We note that similar results can be obtained for the d-dimensional version of a hafnian, cf. [BS11]. We also note that there is a Bregman-Minc type upper bound, cf. Sect. 3.4, for d-dimensional permanents of 0–1 tensors [DG87].

The entropy-based method of proof of the Bregman-Minc inequality found in [Ra97] (see Sect. 3.4) was further applied to obtain non-trivial upper bounds for the number of independent sets in graphs [Ka01], the number of Hamiltonian cycles in graphs [CK09] and hypergraphs of particular types [LL13], [LL14]. In contrast, lower bounds are usually much harder to come by. A recent breakthrough by Keevash [Ke14], [Ke15] establishes the existence and the asymptotic of the number of designs, which can be interpreted as a result on a lower bound for multidimensional permanents for some special (very symmetric) arrays with 0–1 entries, see also [Po15] for lower bounds complementing [LL13] and [LL14].

Efficient algorithms for computing the d-dimensional permanent exactly in special cases are discussed in [CP16].

4.5 Mixed Discriminants

4.5.1 Definition. Let $Q_1, \ldots, Q_n$ be $n \times n$ real symmetric matrices. Then

$$p(x_1, \ldots, x_n) = \det(x_1 Q_1 + \ldots + x_n Q_n)$$

is a homogeneous polynomial of degree n and its mixed term

$$\frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(x_1, \ldots, x_n) = D(Q_1, \ldots, Q_n)$$

is called the *mixed discriminant* of $Q_1, \ldots, Q_n$. Mixed discriminants were introduced by A.D. Alexandrov in his work on mixed volumes [Al38], see also [Le93].

We can express the mixed discriminant as a polynomial in the entries of the matrices $Q_1, \ldots, Q_n$ as follows: suppose that $Q_k = \left(q_{ij}^k\right)$ for $1 \leq i, j \leq n$ and $k = 1, \ldots, n$. Then

$$x_1 Q_1 + \ldots + x_n Q_n = \left(x_1 q_{ij}^1 + \ldots + x_n q_{ij}^n\right) \quad \text{for} \quad 1 \leq i, j \leq n$$

and hence

$$\det(x_1 Q_1 + \ldots + x_n Q_n) = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \prod_{i=1}^{n} \left(x_1 q^1_{i\sigma(i)} + \ldots + x_n q^n_{i\sigma(i)} \right)$$

and, consequently,

$$D(Q_1, \ldots, Q_n) = \sum_{\sigma, \tau \in S_n} (\operatorname{sgn} \sigma) \prod_{i=1}^{n} q^{\tau(i)}_{i\sigma(i)} \tag{4.5.1.1}$$

Thus the mixed discriminant $D(Q_1, \ldots, Q_n)$ can be interpreted as a version of the determinant of an $n \times n \times n$ array whose 2-dimensional slices are identified with the matrices $Q_1, \ldots, Q_n$, cf. Sect. 4.4.

As follows by (4.5.1.1), the mixed discriminant is linear in each argument. It is immediate from the definition that if T is an $n \times n$ matrix then

$$D\left(T^* Q_1 T, \ldots, T^* Q_n T\right) = (\det T)^2 \, D(Q_1, \ldots, Q_n), \tag{4.5.1.2}$$

where T^* is the transpose of T.

In general, we obtain the monomial expansion

$$\det(x_1 Q_1 + \ldots + x_n Q_n) = \sum_{\substack{m_1, \ldots, m_n \geq 0 \\ m_1 + \ldots + m_n = n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!} D\left(\underbrace{Q_1, \ldots, Q_1}_{m_1 \text{ times}}, \ldots, \underbrace{Q_n, \ldots, Q_n}_{m_n \text{ times}} \right). \tag{4.5.1.3}$$

Indeed, it follows from the definition that $D(Q, \ldots, Q) = n! \det Q$ for every $n \times n$ symmetric matrix Q. For $x = (x_1, \ldots, x_n)$, let $Q_x = x_1 Q_1 + \ldots + x_n Q_n$. Then

$$\det Q_x = \frac{1}{n!} D(Q_x, \ldots, Q_x)$$

and we obtain (4.5.1.3) since the mixed discriminant is linear in each argument and symmetric, that is, does not depend on the order of matrices.

Mixed discriminants generalize permanents: given an $n \times n$ matrix $A = \left(a_{ij}\right)$, let us define $n \times n$ symmetric matrices $Q_1, \ldots, Q_n$ by $Q_i = \operatorname{diag}(a_{i1}, \ldots, a_{in})$, that is, Q_i is the diagonal matrix having the i-th row of A on the diagonal. Then

$$\det(x_1 Q_1 + \ldots + x_n Q_n) = \prod_{j=1}^{n} \left(\sum_{i=1}^{n} x_i a_{ij} \right)$$

and hence $D(Q_1, \ldots, Q_n) = \operatorname{per} A$, cf. Sect. 3.2.1.

Just as the permanent of a non-negative matrix is non-negative, the mixed discriminant of positive semidefinite matrices is non-negative.

4.5.2 Lemma. *Suppose that $Q_1, \ldots, Q_n$ are positive semidefinite $n \times n$ matrices. Then*

$$D(Q_1, \ldots, Q_n) \geq 0.$$

Moreover, if $Q_1, \ldots, Q_n$ are positive definite then $D(Q_1, \ldots, Q_n) > 0$.

Proof. Since $D(Q_1, \ldots, Q_n)$ is a continuous function of $Q_1, \ldots, Q_n$, without loss of generality we may assume that $Q_1, \ldots, Q_n$ are positive definite, in which case we prove that $D(Q_1, \ldots, Q_n) > 0$. We proceed by induction on n. The case of $n = 1$ is clear. Suppose that $n > 1$. Since Q_1 is positive definite, we can write $Q_1 = T^*T$ for some invertible $n \times n$ matrix T and then by (4.5.1.2)

$$\begin{aligned} D(Q_1, \ldots, Q_n) =& D\left(T^*T, Q_1, \ldots, Q_n\right) \\ =& (\det T)^2 D\left(I, \left(T^{-1}\right)^* Q_2 T^{-1}, \ldots, \left(T^{-1}\right)^* Q_n T^{-1}\right), \end{aligned} \tag{4.5.2.1}$$

where I is the $n \times n$ identity matrix. For $i = 1, \ldots, n$, let u_i be the matrix having 1 in the i-th diagonal position and zeros elsewhere, so that $I = u_1 + \ldots + u_n$. Denoting

$$Q'_k = \left(T^{-1}\right)^* Q_k T^{-1} \quad \text{for} \quad k = 2, \ldots, n \tag{4.5.2.2}$$

we conclude that $Q'_2, \ldots, Q'_n$ are positive definite matrices and by linearity we have

$$D\left(I, Q'_2, \ldots, Q'_n\right) = \sum_{i=1}^{n} D\left(u_i, Q'_2, \ldots, Q'_n\right). \tag{4.5.2.3}$$

On the other hand, as follows from the definition or from (4.5.1.1), we have

$$D\left(u_i, Q'_2, \ldots, Q'_n\right) = D\left(Q'_{2i}, \ldots, Q'_{ni}\right), \tag{4.5.2.4}$$

where Q'_{ki} is the $(n-1) \times (n-1)$ symmetric matrix obtained from Q'_k by crossing out the ith row and ith column. Since the matrices $Q'_{2i}, \ldots Q'_{ni}$ are positive definite, by the induction hypothesis we conclude that $D\left(Q'_{2i}, \ldots, Q'_{ni}\right) > 0$ and combining (4.5.2.1)– (4.5.2.4), we conclude the proof. □

4.5.3 Combinatorial applications of mixed discriminants. For a vector $u = (u_1, \ldots, u_n)$, we denote by $u \otimes u$ the $n \times n$ matrix whose (i, j)-th entry is $u_i u_j$. Clearly, $u \otimes u$ is positive semidefinite. Various combinatorial applications of mixed discriminants are based on the following formula:

$$D(u_1 \otimes u_1, \ldots, u_n \otimes u_n) = (\det[u_1, \ldots, u_n])^2, \tag{4.5.3.1}$$

where $u_1, \ldots, u_n \in \mathbb{R}^n$ are vectors and $[u_1, \ldots, u_n]$ is the $n \times n$ matrix with columns $u_1, \ldots, u_n$. By continuity, it suffices to check (4.5.3.1) when $u_1, \ldots, u_n$ is a basis of $\mathbb{R}^n$ and then it follows by (4.5.1.2) from the obvious special case when $u_1, \ldots, u_n$ is the standard orthonormal basis of $\mathbb{R}^n$.

The following application is from Chap. V of [BR97]. Let G be a connected (undirected, with no loops or multiple edges) graph with n vertices and m edges, and suppose that the edges are colored with $n-1$ different colors. Let us direct the edges arbitrarily and consider the $n \times m$ incidence matrix $A = (a_{ij})$ of G, where

$$a_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is the beginning of edge } j, \\ -1 & \text{if vertex } i \text{ is the end of edge } j, \\ 0 & \text{otherwise.} \end{cases}$$

Let us remove an arbitrary row of A and let $u_1, \ldots, a_m$ be the columns of the resulting matrix, interpreted as vectors from $\mathbb{R}^{n-1}$. For $k = 1, \ldots, n-1$, let $J_k \subset \{1, \ldots, m\}$ be the set of indices of edges colored into the k-th color and let

$$Q_k = \sum_{j \in J_k} u_j \otimes u_j \quad \text{for} \quad k = 1, \ldots, n-1.$$

Then $Q_1, \ldots, Q_{n-1}$ are positive semidefinite matrices and $D(Q_1, \ldots, Q_{n-1})$ is the number of spanning trees in G having exactly one edge of each color. Indeed, by linearity of the mixed discriminant and (4.5.3.1), we have

$$D(Q_1, \ldots, Q_{n-1}) = \sum_{j_1 \in J_1, \ldots, j_{n-1} \in J_{n-1}} \left(\det\left[u_{j_1}, \ldots, u_{j_{n-1}}\right]\right)^2.$$

As is well-known (see, for example, Chap. 4 of [E+84]), we have

$$\det\left[u_{j_1}, \ldots, u_{j_n}\right] = \begin{cases} \pm 1 & \text{if the edges } j_1, \ldots, j_{n-1} \text{ form a spanning tree in } G \\ 0 & \text{otherwise.} \end{cases}$$

4.5.4 Doubly stochastic n-tuples. Pursuing an analogy with the permanent, we say that the n-tuple $(Q_1, \ldots, Q_n)$ of $n \times n$ positive semidefinite matrices is *doubly stochastic* if

$$\operatorname{tr} Q_1 = \ldots = \operatorname{tr} Q_n = 1 \quad \text{and} \quad Q_1 + \ldots + Q_n = I,$$

the identity matrix. Indeed, if $Q_1, \ldots, Q_n$ are diagonal matrices then $(Q_1, \ldots, Q_n)$ is doubly stochastic if and only if the $n \times n$ matrix A whose i-th row is the diagonal of Q_i is doubly stochastic.

The following result was conjectured by Bapat [Ba89] and proved by Gurvits [Gu06], [Gu08].

4.5.5 Theorem. *Let $(Q_1, \ldots, Q_n)$ be a doubly stochastic n-tuple. Then*

$$D(Q_1, \ldots, Q_n) \geq \frac{n!}{n^n}.$$

The proof follows the approach of Sect. 3.3, which in turn follows [Gu08].

4.5.6 Lemma. *Let $Q_1, \ldots, Q_n$ be $n \times n$ positive definite matrices. Then the polynomial*

$$p\,(x_1, \ldots, x_n) = \det\,(x_1 Q_1 + \ldots + x_n Q_n)$$

is $\mathbb{H}$-stable and the coefficient of every monomial of degree n is positive.

Proof. Let us choose $z_1, \ldots, z_n \in \mathbb{C}$ such that $\Im z_k > 0$ for $k = 1, \ldots, n$ and suppose that $p\,(z_1, \ldots, z_n) = 0$. Then the matrix

$$Q = \sum_{k=1}^{n} z_k Q_k$$

is not invertible and hence there is a vector $y \in \mathbb{C}^n \setminus \{0\}$ such that $Qy = 0$. We consider the standard inner product

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k \overline{y_k} \quad \text{for} \quad x = (x_1, \ldots, x_n) \quad \text{and} \quad y = (y_1, \ldots, y_n)$$

in $\mathbb{C}^n$. Thus we have

$$0 = \langle Qy, y\rangle = \sum_{k=1}^{n} z_k \langle Q_k y, y\rangle. \tag{4.5.6.1}$$

However, since $Q_1, \ldots, Q_n$ are positive definite matrices, the numbers $\langle Q_k y, y\rangle$ are positive real, which contradicts (4.5.6.1) since the imaginary part of each z_k is positive.

Finally, by (4.5.1.3) and Lemma 4.5.2 the coefficient of $x_1^{m_1} \ldots x_n^{m_n}$ in p where $m_1 + \ldots + m_n = n$ is positive. □

Next, we discuss the capacity of p, see also Lemma 3.3.3.

4.5.7 Lemma. *Let $Q_1, \ldots, Q_n$ be a doubly stochastic n-tuple and let*

$$p(x_1, \ldots, x_n) = \det\,(x_1 Q_1 + \ldots + x_n Q_n)\,.$$

Then

$$\inf_{x_1, \ldots, x_n > 0} \frac{p\,(x_1, \ldots, x_n)}{x_1 \cdots x_n} = 1.$$

Proof. Let us define a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$f\,(t_1, \ldots, t_n) = \ln \det\left(e^{t_1} Q_1 + \ldots + e^{t_n} Q_n\right)$$

and let $H \subset \mathbb{R}^n$ be the hyperplane $t_1 + \ldots + t_n = 0$. It suffices to show that the minimum of f on H is attained at $t_1 = \ldots = t_n = 0$. By Lemma 4.5.6 and

Sect. 2.1.1.3, the function f is convex so it suffices to verify that the gradient of f at $t = 0$ is proportional to the vector $(1, \ldots, 1)$.

Since

$$\nabla (\ln \det X) = (X^*)^{-1},$$

denoting

$$S(t) = \sum_{k=1}^{n} e^{t_k} Q_k,$$

we obtain

$$\frac{\partial f}{\partial t_k}\Big|_{t_1=\ldots=t_n=0} = e^{t_k} \operatorname{tr} \left(Q_k S^{-1}(t)\right) \Big|_{t_1=\ldots=t_n=0} = \operatorname{tr}\ Q_k = 1$$

and the proof follows. □

4.5.8 Proof of Theorem 4.5.5**.** The proof follows by Lemmas 4.5.6, 4.5.7 and Corollary 2.4.6. □

There is a notion of *scaling* for n-tuples of positive semidefinite matrices. Just as an $n \times n$ matrix can be scaled to a doubly stochastic matrix, see Theorem 3.5.2, an n-tuple of positive definite matrices can be scaled to a doubly stochastic n-tuple. The following result was obtain by Gurvits and Samorodnitsky [GS02].

4.5.9 Theorem. *Let $Q_1, \ldots, Q_n$ be $n \times n$ positive definite matrices. Then there is a doubly stochastic n-tuple $(B_1, \ldots, B_n)$, an invertible $n \times n$ matrix T and positive reals $\tau_1, \ldots, \tau_n$ such that*

$$Q_k = \tau_k T^* B_k T \quad \textit{for} \quad k = 1, \ldots, n.$$

Proof. As in the proof of Lemma 4.5.7, we consider the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$f\ (t_1, \ldots, t_n) = \ln \det \left(e^{t_1} Q_1 + \ldots + e^{t_n} Q_n\right)$$

and the hyperplane H defined by the equation $t_1 + \ldots + t_n = 0$. It is not hard to see that f attains its minimum on H at some point $(x_1, \ldots, x_n)$ where the gradient of f is proportional to $(1, \ldots, 1)$. As in the proof of Lemma 4.5.7, we obtain that for some real α and

$$S = e^{x_1} Q_1 + \ldots + e^{x_n} Q_n,$$

we have

$$\frac{\partial f}{\partial t_k}\Big|_{t_1=x_1,\ldots,t_n=x_n} = e^{x_k} \operatorname{tr} \left(Q_k S^{-1}\right) = \alpha \quad \text{for} \quad k = 1, \ldots, n. \tag{4.5.9.1}$$

Since

$$n\alpha = \sum_{k=1}^{n} e^{x_k} \operatorname{tr}\left(Q_k S^{-1}\right) = \operatorname{tr}\left(\sum_{k=1}^{n} e^{x_k} Q_k S^{-1}\right) = n,$$

we conclude that

$$\alpha = 1. \tag{4.5.9.2}$$

Since S is positive definite, we can write it as $S = T^*T$ for an invertible $n \times n$ matrix T. We define

$$B_k = e^{x_k} \left(T^{-1}\right)^* Q_k T^{-1} \quad \text{and} \quad \tau_k = e^{-x_k} \quad \text{for} \quad k = 1, \ldots, n.$$

Clearly, $B_1, \ldots, B_n$ are positive definite matrices,

$$Q_k = \tau_k T^* Q_k T \quad \text{for} \quad k = 1, \ldots, n$$

and

$$\sum_{k=1}^{n} B_k = \left(T^*\right)^{-1} \left(\sum_{k=1}^{n} e^{x_k} Q_k\right) T^{-1} = \left(T^*\right)^{-1} S T^{-1} = I.$$

By (4.5.9.1) and (4.5.9.2) we get

$$\operatorname{tr}\ B_k = e^{x_k} \operatorname{tr}\ \left(T^{-1}\right)^* Q_k T^{-1} = e^{x_k} \operatorname{tr}\ Q_k T^{-1} \left(T^{-1}\right)^* = e^{x_k} \operatorname{tr}\ Q_k S^{-1} = 1,$$

which completes the proof. □

In [GS02], Gurvits and Samorodnitsky also discuss scaling of n-tuples of positive *semi*definite matrices.

4.6 A Version of Bregman–Minc Inequalities for Mixed Discriminants

Theorem 4.5.5 is an extension of the van der Waerden inequality from permanents of doubly stochastic matrices, see Sect. 3.3, to mixed discriminants of doubly stochastic n-tuples of matrices. One can ask if there is a version of the Bregman - Minc inequality for mixed discriminants, see Sect. 3.4. Some weak version of such an inequality is suggested in [B16a]. For what follows, it is convenient to associate with an $n \times n$ matrix Q the quadratic form $q : \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$q(x) = \langle Qx, x \rangle \quad \text{for} \quad x \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. We define the eigenvalues, trace and determinant of q as those of Q. Similarly, we define the mixed discriminant $D(q_1, \ldots, q_n)$ of quadratic forms $q_1, \ldots, q_n \longrightarrow \mathbb{R}$ as $D(Q_1, \ldots, Q_n)$, where Q_i is the matrix of q_i. We observe that if we choose a different orthonormal basis in $\mathbb{R}^n$, the matrices Q_i change $Q_i := U^* Q_i U$ for some orthogonal $n \times n$ matrix U, so that the eigenvalues of Q_i and the mixed discriminant $D(Q_1, \ldots, Q_n)$ do not change.

In particular, $q_1, \ldots, q_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a *doubly stochastic* n-tuple of quadratic forms, if the forms $q_1, \ldots, q_n$ are positive semidefinite, $\operatorname{tr} q_i = 1$ for $i = 1, \ldots, n$ and

$$\sum_{i=1}^{n} q_i(x) = \|x\|^2,$$

where $\| \cdot \|$ is the standard Euclidean norm in $\mathbb{R}^n$.

4.6.1 Definition. Given a real $\alpha \geq 1$, we say that an $n \times n$ positive definite matrix Q is α-*conditioned* if

$$\lambda_{\max}(Q) \ \leq \ \alpha \lambda_{\min}(Q),$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are respectively the largest and the smallest eigenvalues of Q. Equivalently, Q is α-conditioned if for the corresponding quadratic form, we have

$$q(x) \ \leq \ \alpha q(y) \quad \text{for all} \quad x, y \in \mathbb{R}^n \quad \text{such that} \quad \|x\| = \|y\| = 1. \qquad (4.6.1.1)$$

An n-tuple $(Q_1, \ldots, Q_n)$ of $n \times n$ positive definite matrices is α-*conditioned* if each matrix Q_k is α-conditioned for $k = 1, \ldots, n$ and

$$q_i(x) \ \leq \ \alpha q_j(x) \quad \text{for all} \quad 1 \leq i, j \leq n \quad \text{and all} \quad x \in \mathbb{R}^n,$$

where $q_1, \ldots, q_n$ are the corresponding quadratic forms.

Definition 4.6.1 extends Definition 3.5.5 from α-conditioned positive matrices to n-tuples of $n \times n$ positive definite matrices. The following result is obtained in [B16a].

4.6.2 Theorem. *Let $(Q_1, \ldots, Q_n)$ be an α-conditioned doubly stochastic n-tuple of positive definite $n \times n$ matrices. Then*

$$D(Q_1, \ldots, Q_n) \ \leq \ n^{\alpha^2} e^{-(n-1)}.$$

Combining Theorems 4.5.5 and 4.6.2, we conclude that for a fixed $\alpha \geq 1$, the mixed discriminant of an α-conditioned doubly stochastic n-tuple of matrices varies within a polynomial in n factor of e^{-n}, just like in the case of permanents of doubly stochastic matrices, cf. Sect. 3.4.6, hafnians of doubly stochastic symmetric matrices (Theorem 4.2.2) and similarly to multidimensional permanents of d-stochastic tensors (Theorem 4.4.8). It would be interesting to find out if in Theorem 4.6.2 we

can just require that the eigenvalues of the matrices $Q_1, \dots, Q_n$ do not exceed α/n (that would have been a true extension of the Bregman - Minc inequality to the mixed discriminant). By and large, the proof follows the same scheme as the proofs of Theorems 4.2.2 and 4.4.8. It proceeds by combining induction and scaling.

To proceed with the induction, we need a way to pass from an n-tuple of $n \times n$ matrices to $(n-1)$-tuple of $(n-1) \times (n-1)$ matrices. We do so by considering a restriction of the quadratic forms onto a subspace. Let $q_1, \dots q_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ be quadratic forms and let $L \subset \mathbb{R}^n$ be a subspace, $\dim L = m$. Then the restrictions $q_i|L : L \longrightarrow \mathbb{R}$ are quadratic forms on L. Since the subspace L inherits the Euclidean structure from $\mathbb{R}^n$, we can define the mixed discriminant $D(q_1|L, \dots, q_m|L)$.

First, we obtain a version of recursive formulas (3.1.1.2), (4.1.1.3) and (4.4.1.1).

4.6.3 Lemma. *Let $q_1, \dots, q_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ be quadratic forms and let*

$$q_n(x) = \sum_{i=1}^{n} \lambda_i \langle u_i, x\rangle^2,$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues and $u_1, \dots, u_n$ are the corresponding unit eigenvectors of q_n. Then

$$D(q_1, \dots, q_n) = \sum_{i=1}^{n} \lambda_i D\left(q_1|u_i^{\perp}, \dots, q_{n-1}|u_i^{\perp}\right),$$

where $u_i^{\perp} \subset \mathbb{R}^n$ is the orthogonal complement of u_i.

Proof. Since the mixed discriminant is linear in each argument, it suffices to prove that

$$D\left(q_1, \dots, q_{n-1}, \langle u, x\rangle^2\right) = D\left(q_1|u^{\perp}, \dots, q_{n-1}|u^{\perp}\right) \tag{4.6.3.1}$$

for any unit vector $u \in \mathbb{R}^n$. Let us choose an orthonormal basis in $\mathbb{R}^n$ containing u as the last vector, and let $Q_1, \dots, Q_{n-1}$ be the matrices of $q_1, \dots, q_{n-1}$ in this basis. Then the matrices $Q_1', \dots, Q_{n-1}'$ of the restrictions $q_1|u^{\perp}, \dots, q_{n-1}|u^{\perp}$ are the $(n-1) \times (n-1)$ upper left submatrices of $Q_1, \dots, Q_{n-1}$ while the matrix E_n of $\langle u, x\rangle$ is the matrix whose (n,n)-th entry is 1 and all other entries are 0. It then follows that

$$\frac{\partial}{\partial t_n} \det\left(t_1 Q_1 + \ldots + t_{n-1} Q_{n-1} + t_n E_n\right) = \det\left(t_1 Q_1' + \ldots + t_{n-1} Q_{n-1}'\right)$$

and (4.6.3.1) follows by Definition 4.5.1. □

Next, we show that if we scale an α-conditioned n-tuple of positive definite matrices to a doubly stochastic n-tuple, we get an α^2-conditioned n-tuple of matrices (cf. Lemmas 3.5.6, 4.2.5 and 4.4.7). As we will have to deal with restrictions of quadratic forms, we prove the statement in more generality.

4.6.4 Lemma. *Let $q_1, \dots, q_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ be an α-conditioned n-tuple of positive definite quadratic forms, let $L \subset \mathbb{R}^n$ be an m-dimensional subspace, let $T : L \longrightarrow \mathbb{R}^n$ be a linear transformation such that $\ker T = \{0\}$, let $\tau_1, \dots, \tau_m > 0$ be reals, and let us define quadratic forms $p_1, \dots, p_m : L \longrightarrow \mathbb{R}$ by*

$$p_i(x) = \tau_i q_i(Tx) \quad \text{for} \quad x \in L \quad \text{and} \quad i = 1, \dots, m.$$

Suppose that the m-tuple $(p_1, \dots, p_m)$ is doubly stochastic. Then the m-tuple $(p_1, \dots, p_m)$ is α^2-conditioned.

Proof. Let us define $q : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$q(x) = \sum_{i=1}^{m} \tau_i q_i(x) \quad \text{for} \quad x \in \mathbb{R}^n.$$

Then by (4.6.1.1) the form q is α-conditioned, so

$$\lambda_{\max}(q) \ \leq \ \alpha \lambda_{\min}(q),$$

where $\lambda_{\max}(q)$ and $\lambda_{\min}(q)$ are, respectively, the largest and the smallest eigenvalues of q. For all $x, y \in L$ such that $\|x\| = \|y\| = 1$, we have

$$1 = q(Tx) \ \geq \ \lambda_{\min}(q)\|Tx\|^2 \quad \text{and} \quad 1 = q(Ty) \ \leq \ \lambda_{\max}(q)\|Ty\|^2,$$

from which it follows that

$$\|Tx\|^2 \ \leq \ \alpha \|Ty\|^2 \quad \text{for all} \quad x, y \in L \quad \text{such that} \quad \|x\| = \|y\| = 1. \qquad (4.6.4.1)$$

Using that each quadratic form q_i is α-conditioned, we deduce from (4.6.4.1) that for all $x, y \subset L$ such that $\|x\| = \|y\| = 1$, we have

$$\begin{aligned} p_i(x) = \tau_i q_i(Tx) \ &\leq \ \tau_i \lambda_{\max}(q_i)\|Tx\|^2 \ \leq \ \alpha \tau_i \lambda_{\max}(q_i)\|Ty\|^2 \ \leq \ \alpha^2 \tau_i \lambda_{\min}(q_i)\|Ty\|^2 \\ &\leq \alpha^2 \tau_i q_i(Ty) = \alpha^2 p_i(y) \end{aligned}$$

and hence each quadratic form p_i is α^2-conditioned.

Let us now define quadratic forms $r_i : L \longrightarrow \mathbb{R}$ by $r_i(x) = q_i(Tx)$ for $x \in L$ and $i = 1, \dots, m$. Since the n-tuple $(q_1, \dots, q_n)$ is α-conditioned, we have

$$r_i(x) \ \leq \ \alpha r_j(x) \quad \text{for all} \quad x \in L \quad \text{and all} \quad i, j.$$

Therefore,

$$\operatorname{tr} r_i \ \leq \ \alpha \operatorname{tr} r_j \quad \text{for all} \quad i, j.$$

Since

$$1 = \operatorname{tr}\ p_i = \tau_i \operatorname{tr}\ r_i \quad \text{for} \quad i = 1, \ldots, m,$$

we conclude that

$$\tau_i \ \leq \ \alpha\tau_j \quad \text{for all} \quad 1 \leq i, j \leq m. \tag{4.6.4.2}$$

Since the n-tuple $(q_1, \ldots, q_n)$ is α-conditioned, we deduce from (4.6.4.2) that for all $x \in L$ we have

$$p_i(x) = \tau_i q_i(Tx) \ \leq \ \alpha\tau_j q_i(Tx) \ \leq \ \alpha^2 \tau_j q_j(Tx) = \alpha^2 p_j(x),$$

and the m-tuple $(p_1, \ldots, p_m)$ is α^2-conditioned. □

The last ingredient we need to prove Theorem 4.6.2 is a one-sided version of the inequalities of Lemmas 4.2.6 and 4.4.10.

4.6.5 Lemma. *Let $Q_1, \ldots, Q_n$ be $n \times n$ positive definite matrices such that*

$$\sum_{i=1}^{n} \operatorname{tr}\ Q_i = n.$$

Let $(B_1, \ldots, B_n)$ be a doubly stochastic n-tuple, constructed in Theorem 4.5.9, so that

$$Q_k = \tau_k T^* B_k T \quad \textit{for} \quad k = 1, \ldots, n.$$

Then

$$D(B_1, \ldots, B_n) \ \geq \ D(Q_1, \ldots, Q_n).$$

Proof. Let $Q = Q_1 + \ldots + Q_n$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of Q. Then

$$\det Q = \prod_{i=1}^{n} \lambda_i \ \leq \ \left(\frac{1}{n} \sum_{i=1}^{n} \lambda_i \right)^n = \left(\frac{1}{n} \operatorname{tr}\ Q \right)^n = 1.$$

We have

$$D(Q_1, \ldots, Q_n) = \left(\prod_{k=1}^{n} \tau_k \right) (\det T)^2 D(B_1, \ldots, B_n).$$

In the notation of Theorem 4.5.9,

$$\prod_{k=1}^{n} \tau_k = \exp\left\{ -\sum_{k=1}^{n} x_k \right\} = 1,$$

where $x = (x_1, \ldots, x_k)$ is the minimum point of the function

$$f(t_1, \ldots, t_n) = \ln \det \left(e^{t_1} Q_1 + \ldots + e^{t_n} Q_n\right)$$

on the hyperplane $t_1 + \ldots + t_n = 0$. In addition,

$$\begin{aligned}(\det T)^2 &= \det \left(e^{x_1} Q_1 + \ldots + e^{x_n} Q_n\right) = \exp\{f(x_1, \ldots, x_n)\} \\ &\leq \exp\{f(0, \ldots, 0)\} = \det (Q_1 + \ldots + Q_n) = \det Q \leq 1,\end{aligned}$$

and the proof follows. □

Now we are ready to prove Theorem 4.6.2.

4.6.6 Proof of Theorem 4.6.2. We prove a more general statement:

Let $q_1, \ldots, q_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ be an α-conditioned n-tuple of positive definite quadratic forms, let $L \subset \mathbb{R}^n$ be an m-dimensional subspace, let $T : L \longrightarrow \mathbb{R}^n$ be a linear transformation such that $\ker T = \{0\}$ and let $\tau_1, \ldots, \tau_m > 0$ be reals. Let us define quadratic forms $p_i : L \longrightarrow \mathbb{R}$ by

$$p_i(x) = \tau_i q_i(Tx) \quad \text{for} \quad x \in L \quad \text{and} \quad i = 1, \ldots, m.$$

Suppose that $(p_1, \ldots, p_m)$ is a doubly stochastic m-tuple. Then

$$D(p_1, \ldots, p_m) \leq \exp\left\{-(m-1) + \alpha^2 \sum_{k=2}^{m} \frac{1}{k}\right\}. \tag{4.6.6.1}$$

We obtain Theorem 4.6.2 if $m = n$, $T = I$ is the identity map and $\tau_i = 1$ for $i = 1, \ldots, n$.

We proceed to prove the above statement by induction on m.

If $m = 1$ then $D(p_1) = \det p_1 = 1$ and the statement clearly holds.

Suppose that $m \geq 2$. Let

$$p_m(x) = \sum_{j=1}^{m} \lambda_j \langle u_j, x\rangle^2$$

is the spectral decomposition of p_m, where λ_j are the eigenvalues and u_j are the corresponding unit eigenvectors of p_m. Since $\operatorname{tr} p_m = 1$, we have

$$\sum_{j=1}^{m} \lambda_j = 1. \tag{4.6.6.2}$$

Let $L_j = u_j^{\perp}$ for $j = 1, \ldots, m$. Hence $L_j \subset L$ and $\dim L_j = m-1$. Let $\widehat{p}_{ij} = p_i | L_j$ be the restriction of p_i onto L_j, so $\widehat{p}_{ij} : L_j \longrightarrow \mathbb{R}$ are positive definite quadratic forms. By Lemma 4.6.3,

$$D(p_1, \ldots, p_m) = \sum_{j=1}^{m} \lambda_j D\left(\widehat{p}_{1j}, \ldots, \widehat{p}_{(m-1)j}\right). \tag{4.6.6.3}$$

We note that

$$\text{tr } \widehat{p}_{mj} = \text{tr } p_m - \lambda_j = 1 - \lambda_j \ \geq \ 1 - \frac{\alpha^2}{m},$$

since by Lemma 4.6.4 the quadratic form p_m is α^2-conditioned.

Using that $\widehat{p}_{1j} + \ldots + \widehat{p}_{(m-1)j} = \|x\|^2$ for all $x \in L_j$, we get

$$\sigma_j = \sum_{i=1}^{m-1} \text{tr } \widehat{p}_{ij} = (m-1) - \text{tr } \widehat{p}_{mj} \ \leq \ (m-2) + \frac{\alpha^2}{m}$$
$$\text{for} \quad j = 1, \ldots, m. \tag{4.6.6.4}$$

We define quadratic forms $r_{ij} : L_j \longrightarrow \mathbb{R}$ by

$$r_{ij} = \frac{m-1}{\sigma_j} \widehat{p}_{ij} \quad \text{for} \quad i = 1, \ldots, m-1 \quad \text{and} \quad j = 1, \ldots, m.$$

In particular,

$$\sum_{i=1}^{m-1} \text{tr } r_{ij} = m - 1 \quad \text{for} \quad j = 1, \ldots, m. \tag{4.6.6.5}$$

From (4.6.6.4), we get

$$\begin{aligned} D\left(\widehat{p}_{1j}, \ldots, \widehat{p}_{(m-1)j}\right) &= \left(\frac{\sigma_j}{m-1}\right)^{m-1} D\left(r_{1j}, \ldots, r_{(m-1)j}\right) \\ &\leq \left(1 - \frac{1}{m-1} + \frac{\alpha^2}{m(m-1)}\right)^{m-1} D\left(r_{1j}, \ldots, r_{(m-1)j}\right) \\ &\leq \exp\left\{-1 + \frac{\alpha^2}{m}\right\} D\left(r_{1j}, \ldots, r_{(m-1)j}\right) \quad \text{for} \quad j = 1, \ldots, m. \end{aligned} \tag{4.6.6.6}$$

Let $\left(w_{1j}, \ldots, w_{(m-1)j}\right)$ be the doubly stochastic $(m-1)$-tuple of quadratic forms, $w_{ij} : L_j \longrightarrow \mathbb{R}$, obtained from $r_{1j}, \ldots, r_{(m-1)j}$ by scaling as in Theorem 4.5.9. From (4.6.6.5) and Lemma 4.6.5, we have

$$D\left(w_{1j}, \ldots, w_{(m-1)j}\right) \ \geq \ D\left(r_{1j}, \ldots, r_{(m-1)j}\right)$$

and hence from (4.6.6.6), we get

$$D\left(\widehat{p}_{1j}, \ldots, \widehat{p}_{(m-1)j}\right) \ \leq \ \exp\left\{-1 + \frac{\alpha^2}{m}\right\} D\left(w_{1j}, \ldots, w_{(m-1)j}\right). \tag{4.6.6.7}$$

Now we would like to apply the induction hypothesis to the quadratic forms

$$w_{1j}, \ldots, w_{(m-1)j} : L_j \longrightarrow \mathbb{R}.$$

Since the $(m-1)$-tuple $\left(w_{1j}, \ldots, w_{(m-1)j}\right)$ is obtained by scaling from the $(m-1)$-tuple $\left(r_{1j}, \ldots, r_{(m-1)j}\right)$, there is an invertible linear transformation $S_j : L_j \longrightarrow L_j$ and positive numbers $\mu_{1j}, \ldots, \mu_{(m-1)j}$ such that

$$\begin{aligned} w_{ij}(x) =& \mu_{ij} r_{ij}\left(S_j x\right) = \frac{(m-1)\mu_{ij}}{\sigma_j} \widehat{p}_{ij}(S_j x) = \frac{(m-1)\mu_{ij}}{\sigma_j} p_i(S_j x) \\ =& \frac{(m-1)\mu_{ij}\tau_i}{\sigma_j} q_i(T S_j x) \quad \text{for all} \quad x \in L_j \end{aligned}$$

and $i = 1, \ldots, m-1$. For each $j = 1, \ldots, m$, we have a linear transformation $T S_j : L_j \longrightarrow \mathbb{R}^n$ with $\ker T S_j = \{0\}$ and hence by the induction hypothesis

$$D\left(w_{1j}, \ldots, w_{(m-1)j}\right) \;\leq\; \exp\left\{-(m-2) + \alpha^2 \sum_{k=2}^{m-1} \frac{1}{k}\right\} \tag{4.6.6.8}$$

for $j = 1, \ldots, m$.

Combining (4.6.6.2), (4.6.6.3), (4.6.6.7) and (4.6.6.8), we obtain (4.6.6.1), which completes the proof. □

4.6.7 Computing mixed discriminants. If the n-tuple $(Q_1, \ldots, Q_n)$ is a doubly stochastic then by Lemma 4.5.6 we have $D(Q_1, \ldots, Q_n) \leq \det(Q_1 + \ldots + Q_n) = 1$. This, together with Theorem 4.5.5, the scaling algorithm of Theorem 4.5.9 and the formula

$$D\left(\lambda_1 T^* B_1 T, \ldots, \lambda_n T^* B_n T\right) = \left(\prod_{k=1}^{n} \lambda_k\right) (\det T)^2 \, D(B_1, \ldots, B_n) \tag{4.6.7.1}$$

results in a deterministic polynomial time algorithm to approximate the mixed discriminant $D(Q_1, \ldots, Q_n)$ of positive semidefinite matrices within a multiplicative factor of $n!/n^n \approx e^{-n}$ [GS02]. A better approximation factor can be achieved by a randomized polynomial time algorithm [Ba99], extending the permanent approximation algorithm of Sect. 3.9.1. Namely, given $n \times n$ positive semidefinite matrices $Q_1, \ldots, Q_n$, we compute $n \times n$ matrices $T_1, \ldots, T_n$ such that $Q_k = T_k^* T_k$ for $k = 1, \ldots, n$. Let $u_1, \ldots, u_n$ be vectors sampled independently at random from the standard Gaussian distribution in $\mathbb{R}^n$ and let $[T_1 u_1, \ldots, T_n u_n]$ be the $n \times n$ matrix with columns $T_1 u_1, \ldots, T_n u_n$. Using formula (4.5.3.1) it is not hard to show that

$$D(Q_1, \ldots, Q_n) = \mathbf{E}\left(\det[T_1 u_1, \ldots, T_n u_n]\right)^2,$$

and that with probability approaching 1 as n grows we have

$$\det\left([T_1u_1,\ldots,T_nu_n]\right)^2 \;\geq\; (0.28)^n D\left(Q_1,\ldots,Q_n\right).$$

If vectors $u_1,\ldots,u_n$ are sampled independently at random from the standard Gaussian distribution in $\mathbb{C}^n$ then

$$D\left(Q_1,\ldots,Q_n\right) = \mathbf{E}\,\left|\det\left[T_1u_1,\ldots,T_nu_n\right]\right|^2,$$

and that with probability approaching 1 as n grows we have

$$\left|\det\left[T_1u_1,\ldots,T_nu_n\right]\right|^2 \;\geq\; (0.56)^n D\left(Q_1,\ldots,Q_n\right).$$

Finally, assume that $u_1,\ldots,u_n$ are sampled from the standard Gaussian distribution in the quaternionic space $\mathbb{H}^n$ and let $[Tu_1,\ldots,Tu_n]_{\mathbb{C}}$ be $2n\times 2n$ complex matrix constructed from the $n\times n$ quaternionic matrix $[Tu_1,\ldots,Tu_n]$ as in Sect. 3.9.1. Then $\det\left[Tu_1,\ldots,Tu_n\right]_{\mathbb{C}}$ is a non-negative real,

$$\mathbf{E}\,\det\left[Tu_1,\ldots,Tu_n\right]_{\mathbb{C}} = D\left(Q_1,\ldots,Q_n\right)$$

and with probability approaching 1 as n grows, we have

$$\det\left[Tu_1,\ldots,Tu_n\right]_{\mathbb{C}} \;\geq\; (0.76)^n D\left(Q_1,\ldots,Q_n\right).$$

Assume now that the n-tuple $(Q_1,\ldots,Q_n)$ is α-conditioned. As follows from Theorem 4.6.2 and Lemma 4.6.4, the scaling algorithm of Theorem 4.5.9, together with formula (4.6.7.1) and Theorem 4.5.5, approximates the mixed discriminant $D\left(Q_1,\ldots,Q_n\right)$ within a factor of $n^{O(\alpha^2)}$, which is polynomial in n provided α is fixed in advance, cf. also Sects. 3.5, 4.2 and 4.4.

In their proof of the Kadison–Singer Conjecture [M+15], Marcus, Spielman and Srivastava introduce and study the *mixed characteristic polynomial*

$$p_{Q_1,\ldots,Q_n}(x) = \prod_{i=1}^{n}\left(1-\frac{\partial}{\partial z_i}\right)\det\left(xI+\sum_{i=1}^{n} z_i\,Q_i\right)\Big|_{z_1=\ldots=z_n=0},$$

where $Q_1,\ldots,Q_n$ are real symmetric or complex Hermitian $m\times m$ matrices. If $Q_1,\ldots,Q_n$ are positive semidefinite then the roots of the mixed characteristic polynomial are real and necessarily non-negative. If $m=n$ then the constant term of $p_{Q_1,\ldots,Q_n}$, up to a sign, is equal to the mixed discriminant $D(Q_1,\ldots,Q_n)$. The relation of the mixed characteristic polynomial to the mixed discriminant is similar to the relation of the matching polynomial of Chap. 5 to the permanent and hafnian.

Chapter 5
The Matching Polynomial

Known in statistical physics as the partition function of the monomer-dimer model, the matching polynomial of a graph is an extension of the hafnian, as it enumerates all, not necessarily perfect, matchings in the graph. The Heilmann–Lieb Theorem asserts that the roots of the matching polynomial (with non-negative real weights on the edges) are negative real, which allows us to efficiently approximate the polynomial through interpolation anywhere away from the negative real axis. We demonstrate the "correlation decay" phenomenon of the probability for a random matching to contain a given vertex to be asymptotically independent on whether the matching contains some other remote vertex. Through the Csikvári–Lelarge "lifting" argument, it allows us to lower bound the matching polynomial of a bipartite graph by the matching polynomial of a covering tree, which produces a useful Bethe-entropy estimate. Finally, we prove a general bound on the complex roots of the hypergraph matching polynomial, which allows us to obtain new interpolation results for (multidimensional) permanents of matrices and tensors that are not very far from the matrices (tensors) of all 1 s in the ℓ^1 distance on the slices.

5.1 Matching Polynomial

5.1.1 Definition. Let $A = \left(a_{ij}\right)$ be an $n \times n$ symmetric matrix. For a positive integer m such that $2m \le n$, we define

$$\operatorname{haf}_m(A) = \sum_{\{i_1 j_1\},\ldots,\{i_m, j_m\}} a_{i_1 j_1} \cdots a_{i_m j_m}, \tag{5.1.1.1}$$

where the sum is taken over all unordered collections of m pairwise disjoint unordered pairs $\{i_1, j_1\}, \ldots, \{i_m, j_m\}$ where $1 \le i_1, j_1, \ldots, i_m, j_m \le n$. In particular, if n is even and $n = 2m$ then $\operatorname{haf}_m(A) = \operatorname{haf} A$. We also agree that $h_0(A) = 1$. Thus $h_m(A)$

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enumerates all matchings consisting of m edges in a complete weighted graph with n vertices.

We define the univariate *matching polynomial* by

$$p_A(t) = \sum_{m=0}^{\lfloor n/2 \rfloor} \mathrm{haf}_m(A) t^m.$$

In statistical physics, $p_A(t)$ is known as the partition function of the "monomer-dimer model", where edges of the matching correspond to "dimers" while the vertices of the graph not covered by the matching correspond to single "atoms".

The following remarkable result was obtained by Heilmann and Lieb [HL72].

5.1.2 Theorem. *Let A be an $n \times n$ symmetric matrix with non-negative entries and let*

$$\beta = \beta_A = \max_{i=1,\dots,n} \sum_{j:j\neq i} a_{ij}.$$

Then the roots of the matching polynomial $p_A(t)$ are negative real and satisfy the inequality

$$t \leq -\frac{1}{4\beta}.$$

The bound on the roots obtained in [HL72] is, in fact, slightly better, cf. Remark 5.1.4 below.

We follow [HL72] and deduce Theorem 5.1.2 from the following result.

5.1.3 Theorem. *For a symmetric $n \times n$ matrix A let us define a univariate polynomial*

$$q_A(t) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m h_m(A) t^{n-2m}.$$

(1) Suppose that A is a real symmetric matrix with positive off-diagonal entries and let A_i be the $(n-1) \times (n-1)$ matrix obtained from A by crossing out the i-th row and i-th column of A for some $i = 1, \dots, n$. Then the roots of $q_A(t)$ and $q_{A_i}(t)$ are real and $q_{A_i}(t)$ interlaces $q_A(t)$ provided $n \geq 2$.

(2) Suppose that A is a non-negative real matrix. Then the roots of $q_A(t)$ are real.

(3) Let A be an $n \times n$ symmetric non-negative real matrix and let

$$\beta = \beta_A = \max_{i=1,\dots n} \sum_{j:j\neq i} a_{ij}.$$

If $q_A(t) = 0$ then $|t| \leq 2\sqrt{\beta}$.

Proof. To prove Part (1), we proceed by induction on $n = \deg q_A$. If $n = 2$ and $i \in \{1, 2\}$ we have

$$q_A(t) = t^2 - a_{12} \quad \text{and} \quad q_{A_i}(t) = t,$$

and hence $q_{A_i}(t)$ indeed interlaces $q_A(t)$.

Suppose that $n > 2$. We split all matchings in the complete graph with vertices $\{1, \dots, n\}$ contributing to (5.1.1.1) into two classes: those that contain i and those that do not. Then we obtain the recurrence relation:

$$q_A(t) = tq_{A_i}(t) - \sum_{j:\ j \neq i} a_{ij} q_{A_{ij}}(t), \tag{5.1.3.1}$$

where A_{ij} is the $(n-2) \times (n-2)$ symmetric matrix obtained from A by crossing out the i-th and j-th row and the i-th and j-th column. The polynomial $q_{A_i}(t)$ in (5.1.3.1) accounts for the matchings not containing i while the sum in (5.1.3.1) accounts for the matchings containing i. We note that the highest terms of $q_A(t)$, $q_{A_i}(t)$ and $q_{A_{ij}}(t)$ are positive (with coefficients equal to 1).

By the induction hypothesis, each $q_{A_{ij}}(t)$ interlaces $q_{A_i}(t)$ and hence by Part (1) of Theorem 2.3.2, the polynomial

$$p(t) = \sum_{j:j \neq i} a_{ij} q_{A_{ij}}(t)$$

interlaces $q_{A_i}(t)$. Then by Part (2) of Theorem 2.3.2, the polynomial $q_{A_i}(t)$ interlaces $q_A(t) = tq_{A_i}(t) - p(t)$.

As follows by Part (1), the roots of $q_A(t)$ are real if A is a symmetric real matrix with positive off-diagonal entries. It then follows by continuity that the roots of $q_A(t)$ are real if A is a non-negative real matrix, which proves Part (2).

To prove Part (3), we may assume that $\beta > 0$ since the case of $\beta = 0$ is trivial. For a subset $I \subset \{1, \dots, n\}$ we denote by A_I the submatrix of A obtained from A by crossing out rows and columns in I. We denote $q_{A_I}(t)$ just by $q_I(t)$ and prove by descending induction on $|I| = n-2, n-3, \dots, 0$ that

$$q_I(t) \neq 0 \quad \text{and} \quad \frac{q_I(t)}{q_{I \cup \{i\}}(t)} \geq \sqrt{\beta} \quad \text{provided} \quad i \notin I \quad \text{and} \quad t \geq 2\sqrt{\beta}.$$

Indeed, if $I = \{1, \dots, n\} \setminus \{i, j\}$, we have

$$q_I(t) = t^2 - a_{ij}, \quad q_{I \cup \{i\}}(t) = t \quad \text{and} \quad \frac{q_I(t)}{q_{I \cup \{i\}}(t)} = t - \frac{a_{ij}}{t} \geq 2\sqrt{\beta} - \frac{1}{2}\sqrt{\beta} \geq \sqrt{\beta}$$

provided $t \geq 2\sqrt{\beta}$.

If $|I| < n-2$, using (5.1.3.1), for all $i \notin I$ we can write

$$q_I(t) = tq_{I \cup \{i\}}(t) - \sum_{\substack{j:j \notin I \\ j \neq i}} a_{ij} q_{I \cup \{i,j\}}(t) \tag{5.1.3.2}$$

and hence

$$\frac{q_I(t)}{q_{I\cup\{i\}}(t)} = t - \sum_{\substack{j:j\notin I\\ j\neq i}} a_{ij}\frac{q_{I\cup\{i,j\}}(t)}{q_{I\cup\{i\}}(t)}.$$

By the induction hypothesis, for $t \geq 2\sqrt{\beta}$ we have

$$\frac{q_{I\cup\{i,j\}}(t)}{q_{I\cup\{i\}}(t)} \leq \frac{1}{\sqrt{\beta}}$$

and hence

$$\frac{q_I(t)}{q_{I\cup\{i\}}(t)} \geq 2\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\sum_{\substack{j:j\notin I\\ j\neq i}} a_{ij} \geq 2\sqrt{\beta} - \sqrt{\beta} = \sqrt{\beta},$$

which completes the induction. Hence we proved that $q_A(t) \neq 0$ provided $t > 2\sqrt{\beta}$. Since the polynomial $q_A(t)$ is even when n is even and odd when n is odd, the proof follows. □

5.1.4 Remark. In [HL72] a slightly stronger bound is proven (by a more careful induction): let us define

$$\begin{aligned} w_i &= \left(\sum_{j:j\neq i} a_{ij}\right) - \min_{\substack{j:j\neq i\\ a_{ij}>0}} a_{ij}\\ \beta_1 &= \max_{i=1,\ldots,n} w_i\\ \beta_2 &= \frac{1}{4}\max_{i,j} a_{ij}\\ \beta &= \max\{\beta_1, \beta_2\}. \end{aligned}$$

Then $q_A(t) \neq 0$ for $|t| \geq 2\sqrt{\beta}$. In particular, if A is the adjacency matrix of a graph G with maximum degree $\Delta(G) > 1$ of a vertex, we have $q_A(t) \neq 0$ for $|t| \geq 2\sqrt{\Delta(G)-1}$.

5.1.5 Proof of Theorem 5.1.2. Let $q_A(t)$ be the polynomial of Theorem 5.1.3. Then

$$q_A(t) = t^n p_A\left(-\frac{1}{t^2}\right).$$

By Part (2) of Theorem 5.1.3 it follows that the roots of $p_A(t)$ are the (necessarily real negative) numbers $-1/t^2$ where t are non-zero roots of $q_A(t)$. Since by Part (3) of Theorem 5.1.3, every real root t of $q_A(t)$ satisfies $|t| \leq 2\sqrt{\beta}$, we conclude that all roots of $p_A(t)$ satisfy $t \leq -1/4\beta$, as desired. □

One immediate corollary is that the numbers $\mathrm{haf}_m(A)$ form a log-concave sequence.

5.1.6 Corollary. *Let $A = (a_{ij})$ be a non-negative symmetric matrix. Then*

$$(\mathrm{haf}_m(A))^2 \geq \mathrm{haf}_{m-1}(A)\,\mathrm{haf}_{m+1}(A) \quad \textit{for} \quad m = 1, \ldots, \lfloor n/2 \rfloor - 1.$$

Proof. Follows by Theorem 5.1.2 and Theorem 2.3.3. □

5.1.7 Computing the matching polynomial. Let A be an $n \times n$ non-negative real symmetric matrix, let $p_A(t)$ be the corresponding matching polynomial and let $\beta = \beta_A \geq 0$ be as defined in Theorem 5.1.2. Let us fix some $0 < \delta < 1$. One can deduce from Theorem 5.1.2 that for any given $0 < \epsilon < 1$ and complex t the value of $p_A(t)$ can be approximated within a relative error of ϵ in quasi-polynomial $n^{O(\ln n - \ln \epsilon)}$ time as long as $|t| \leq \delta/4\beta$, and, moreover, $\ln p_A(t)$ can be approximated within an additive error $\epsilon > 0$ by a polynomial of degree $O(\ln n - \ln \epsilon)$ in t and the entries of A. Given such a t, we define a univariate polynomial

$$g_A(z) = p_A(tz).$$

From Theorem 5.1.2, we deduce that $g(z) \neq 0$ as long as $|z| \leq 1/\delta$. We define

$$f_A(z) = \ln g_A(z)$$

and use Lemma 2.2.1 to approximate $f_A(1) = \ln p_A(t)$ by the Taylor polynomial of $f_A(z)$ at $z = 0$ of some degree $d = O(\ln n - \ln \epsilon)$. Since the values of $\mathrm{haf}_m(A)$ can be computed exactly in $n^{O(m)}$ time, we can compute the m-th derivative of $p_A(t)$ at $t = 0$ in $n^{O(m)}$ time and hence the m-th derivative of $f_A(z)$ at $z = 0$ in $n^{O(m)}$ time, cf. Sect. 2.2.2, see also [Re15].

In fact, for *any* $\delta > 1$, fixed in advance, the value of $p_A(t)$ at a complex t can be approximated within a relative error $0 < \epsilon < 1$ in $n^{O(\ln n - \ln \epsilon)}$ time as long as

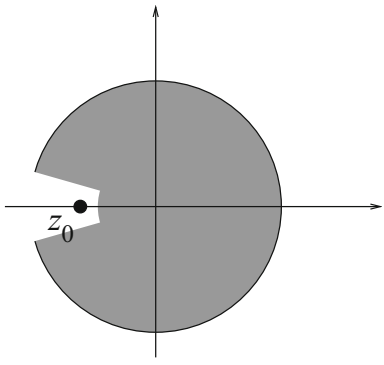


Fig. 5.1 The region where $p_A(t)$ can be efficiently approximated; z_0 is an upper bound on the roots of $p_A(t)$

$$|t| \leq \frac{\delta}{4\beta} \quad \text{and} \quad |\pi - \arg t| \geq \frac{1}{\delta}.$$

Moreover, in that region $\ln p_A(t)$ can be approximated within an error ϵ by a polynomial of degree $O(\ln n - \ln \epsilon)$ in t and the entries of A. Figure 5.1 shows a domain of $t \in \mathbb{C}$ for which $p_A(t)$ can be approximated in quasi-polynomial time. It consists of an outer disc of radius $\delta|z_0|$ for some fixed $\delta > 1$, where z_0 is an upper bound on the roots of p_A, with a sector of a fixed angle removed, and an inner disc of radius $\gamma|z_0|$ for some fixed $0 < \gamma < 1$.

To approximate $p_A(t)$, using Lemma 2.2.3 we first construct a disc $D = \{z \in \mathbb{C} : |z| \leq \beta\}$ of some radius $\beta = \beta(\delta) > 1$ and a polynomial $\psi = \psi_\delta : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\psi(0) = 0$, $\psi(1) = 1$ and the image $\psi(D)$ lies in a sufficiently thin strip aligned with the positive real axis, so that the set $t\psi(D)$ does not contain the roots of p_A. We then consider the composition

$$g_A(z) = p_A(t\psi(z))$$

and use the Taylor polynomial of $f_A(z) = \ln g_A(z)$ at $z = 0$ to approximate $g_A(1) = p_A(t)$, cf. Sect. 3.7 and see [PR16] for detail.

Patel and Regts further showed [PR16] that if A is the adjacency matrix of a graph G with the largest degree $\Delta(G)$ of a vertex bounded above in advance, then the above algorithm for approximating $p_A(t)$ can be made polynomial and not just quasi-polynomial. They show that in that case the values of $\mathrm{haf}_m(A)$ for $m = O(\ln n - \ln \epsilon)$ can be computed in time polynomial in n and $1/\epsilon$, see also Sect. 6.6.

Let z_0 be the largest root of the matching polynomial $p_A(t)$ (since $z_0 < 0$, it is also the root of $p_A(t)$ nearest to the origin). Then $\pm 1/\sqrt{-z_0}$ are the roots of the polynomial $q_A(t)$ of Theorem 5.1.3 of the largest absolute value. We note that $q_A(t)$ is a monic polynomial and that the coefficient $h_m(A)$ of t^{n-2m} can be computed in $n^{O(2m)}$ time simply by enumerating all matchings of size m in the complete graph. Arguing as in Sect. 2.3.4, we can estimate the largest absolute value of the root of $q_A(t)$ and hence the value of z_0 within relative error ϵ in $n^{O(\ln n - \ln \epsilon)}$ time.

There is a Markov Chain based randomized polynomial time algorithm approximating $p_A(t)$ for real $t \geq 0$, see Chap. V of [Je03]. If A is the adjacency matrix of a graph, the complexity of the algorithm is polynomial in t.

For zeros of partition functions of subgraphs with various degree constraints, see [Ru99, Wa99].

5.2 Correlation Decay for the Matching Polynomial

5.2.1 Graphs and probabilities. In what follows, it is convenient to switch from the language of symmetric matrices to the language of weighted graphs. We consider a graph $G = (V, E; A)$, undirected, without loops or multiple edges, with set V of

vertices, set E of edges and non-negative weights $a_e : e \in E$ on the edges. We define the matching polynomial

$$P_G(t) = \sum_{k=0}^{\lfloor |V|/2 \rfloor} h_k t^k, \quad \text{where} \quad h_k = \sum_{\substack{e_1,\dots,e_k \in E: \\ e_1,\dots,e_k \text{ pairwise disjoint}}} a_{e_1} \cdots a_{e_k}.$$

We call the product $a_{e_1} \cdots a_{e_k}$ the *weight* of a matching $e_1, \dots, e_k$.

When G is the complete graph with set $V = \{1, \dots, n\}$ of vertices and weights $a_e = a_{ij}$ for $e = \{i, j\}$, where $A = \left(a_{ij}\right)$ is a symmetric non-negative matrix, we obtain the matching polynomial $p_A(t)$ of Sect. 5.1.1.

We assume that the parameter t is non-negative real. Let us consider the set of all matchings in G as a finite probability space, where the probability of a matching consisting of the edges $e_1, \dots, e_k$ is proportional to $t^k a_{e_1} \cdots e_k$ (if $k = 0$ we assume that the product is equal to 1). Then the probability that a random matching contains k edges is $t^k h_k / P_G(t)$ and

$$\frac{t P'_G(t)}{P_G(t)} = t \frac{d}{dt} \ln P_G(t) = P_G^{-1}(t) \sum_{k=0}^{\lfloor |V|/2 \rfloor} k h_k t^k$$

is the expected number of edges in a random matching.

Let $G = (V, E; A)$ be a weighted graph as above, and let $S \subset V$ be a set of its vertices. We denote by $G - S$ the weighted graph obtained from G by deleting all vertices from S together with incident edges. We start with a recurrence relation similar to (5.1.3.1):

$$P_G(t) = P_{G-v}(t) + t \sum_{\substack{w \in V: \\ \{w,v\} \in E}} a_{\{v,w\}} P_{G-v-w}(t) \tag{5.2.1.1}$$

Here v is a vertex of V, the term $P_{G-v}(t)$ enumerates all matchings in G not containing v whereas the sum accounts for all matchings in G containing v (we use $G - v$ as a shorthand for $G - \{v\}$ and $G - v - w$ as a shorthand for $G - \{v, w\}$) We rewrite (5.2.1.1) as

$$\frac{P_{G-v}(t)}{P_G(t)} = \left(1 + t \sum_{\substack{w \in V: \\ \{w,v\} \in E}} a_{\{v,w\}} \frac{P_{G-v-w}(t)}{P_{G-v}(t)} \right)^{-1}. \tag{5.2.1.2}$$

We note that $P_{G-v}(t)/P_G(t)$ is the probability that a random matching does not contain vertex v whereas $P_{G-v-w}(t)/P_{G-v}(t)$ is the conditional probability that a random matching does not contain vertex w given that it does not contain vertex v. We note that the sum

$$\frac{1}{2}\sum_{v\in V}\left(1-\frac{P_{G-v}(t)}{P_G(t)}\right)$$

represents the expected number edges (half of the expected number of vertices) in a random matching, and hence we get

$$t\frac{d}{dt}\ln P_G(t)=\frac{1}{2}\sum_{v\in V}\left(1-\frac{P_{G-v}(t)}{P_G(t)}\right). \tag{5.2.1.3}$$

Formula (5.2.1.2) can be naturally generalized as follows: for a set $S\subset V$ of vertices and a vertex $v\in V\setminus S$, we have

$$\frac{P_{G-S-v}(t)}{P_{G-S}(t)}=\left(1+t\sum_{\substack{w\in V\setminus S:\\ \{w,v\}\in E}}a_{\{v,w\}}\frac{P_{G-S-v-w}(t)}{P_{G-S-v}(t)}\right)^{-1}. \tag{5.2.1.4}$$

We interpret $P_{G-S-v}(t)/P_{G-S}(t)$ as the conditional probability that a random matching in G does not contain vertex v, given that it does not containing vertices from S.

We discuss a dynamic programming type algorithm for computing the probabilities $P_{G-S-v}(t)/P_{G-S}(t)$ and, as a corollary, the matching polynomial $P_G(t)$, which exhibits an interesting phenomenon, called the "correlation decay". We follow [B+07] with some modifications.

5.2.2 Lemma. *Let us consider the set X of all non-negative vectors $x=\left(x_{S,v}\right)$ with coordinates parameterized by a pair consisting of a set $S\subset V$ of vertices and a vertex $v\in V\setminus S$ and let us define a transformation $T: X\longrightarrow X$ by*

$$T(x)=y \quad \textit{where} \quad y_{S,v}=\left(1+t\sum_{\substack{w\in V\setminus S:\\ \{w,v\}\in E}}a_{\{v,w\}}x_{\{S,v\},w}\right)^{-1}.$$

Let

$$\beta=\max_{v\in V}\sum_{\substack{w\in V:\\ \{v,w\}\in E}}a_{\{v,w\}}$$

and suppose that

$$t=\frac{\lambda}{\beta} \quad \textit{for some} \quad \lambda>0.$$

(1) Suppose that

$$\frac{1}{1+\lambda} \;\le\; x_{S,v} \;\le\; 1 \quad \textit{for all} \quad S \subset V \quad \textit{and} \quad v \in V \setminus S.$$

Then for $y = T(x)$ *we have*

$$\frac{1}{1+\lambda} \;\le\; y_{S,v} \;\le\; 1 \quad \textit{for all} \quad S \subset V \quad \textit{and} \quad v \in V \setminus S.$$

(2) For any $x', x'' \in X$ *and* $y' = T(x')$, $y'' = T(x'')$, *we have*

$$\max_{\substack{S\subset V,\\ v\in V\setminus S}} \left|\ln y'_{S,v} - \ln y''_{S,v}\right| \;\le\; \frac{\lambda}{\lambda+1} \max_{\substack{S\subset V,\\ v\in V\setminus S}} \left|\ln x'_{S,v} - \ln x''_{S,v}\right|.$$

Proof. Since $x_{S,v} \ge 0$ for all S and v, for $y = T(x)$ we have $y_{S,v} \le 1$ for all S and v. If, in addition, $x_{S,v} \le 1$ for all S and v then

$$t \sum_{\substack{w\in V\setminus S:\\ \{w,v\}\in E}} a_{\{v,w\}} x_{\{S,v\},w} \;=\; \frac{\lambda}{\beta} \sum_{\substack{w\in V\setminus S:\\ \{w,v\}\in E}} a_{\{v,w\}} \;\le\; \lambda$$

and $y_{S,v} \ge (1+\lambda)^{-1}$ for all S and v, which proves Part (1).

To prove Part (2), we introduce the substitution

$$\xi_{S,v} = -\ln x_{S,v} \quad \text{and} \quad \eta_{S,v} = -\ln y_{S,v}.$$

Then the transformation T is written as

$$\eta_{S,v} = \ln\left(1 + t \sum_{\substack{w\in V\setminus S:\\ \{w,v\}\in E}} a_{\{v,w\}} e^{-\xi_{\{S,v\},w}}\right).$$

Then

$$\sum_{\substack{w\in V\setminus S\\ \{w,v\}\in E}} \left|\frac{\partial \eta_{S,v}}{\partial \xi_{\{S,v\},w}}\right| = \left| \frac{t \displaystyle\sum_{\substack{w\in V\setminus S:\\ \{w,v\}\in E}} a_{\{v,w\}} e^{-\xi_{\{S,v\},w}}}{1 + t \displaystyle\sum_{\substack{w\in V\setminus S:\\ \{w,v\}\in E}} a_{\{v,w\}} e^{-\xi_{\{S,v\},w}}} \right|$$

$$= \left| 1 - \frac{1}{1 + t \sum\limits_{\substack{w \in V \setminus S: \\ \{w,v\} \in E}} a_{\{v,w\}} e^{-\xi_{\{S,v\},w}}} \right|$$
$$\leq \left| 1 - \frac{1}{1+\lambda} \right| = \frac{\lambda}{1+\lambda}$$

and the proof of Part (2) follows. □

5.2.3 Correlation decay. The transformation T of Lemma 5.2.2 is a *contraction*. If we start with a vector $x_{S,v} = 1$ (or any other vector $(1+\lambda)^{-1} \leq x_{S,v} \leq 1$) and iterate T, then the vector $T^m(x)$ necessarily converges to the unique fixed point x^* of T, which, by (5.2.1.4) necessarily satisfies

$$x^*_{S,v} = \frac{P_{G-S-v}(t)}{P_{G-S}(t)}.$$

As follows from Lemma 5.2.2, to approximate x^* by $T^m(x)$ coordinate-wise within a relative error $0 < \epsilon < 1$, we can choose

$$m = O\left(\frac{\ln 1/\epsilon}{\ln(\lambda+1) - \ln \lambda} \right) \tag{5.2.3.1}$$

iterations.

Let us introduce a metric on the set V of vertices, where $\mathrm{dist}(u, v)$ is the smallest possible number of edges of G in a path connecting u and v (we let $\mathrm{dist}(u, v) = +\infty$ if vertices u and v lie in different connected components). We note that to compute the (S, v)-coordinate of $T^m(x)$, we only need to access (S', w')-coordinates, where $\mathrm{dist}(u, v) \leq m$ for all $u \in \left(S' \setminus S\right) \cup \{w'\}$. As follows from (5.2.3.1), if λ is fixed in advance, we obtain a quasi-polynomial algorithm of $|V|^{O(\ln|V| - \ln\epsilon)}$ complexity to approximate $x^*_{S,v}$ within relative error $0 < \epsilon < 1$.

In particular, if we fix some $\lambda_0 > 0$ and $\epsilon > 0$ then for any $\lambda \leq \lambda_0$, up to an additive error ϵ, the value of $P_{G-\{v\}}(t)/P_G(t)$ depends only on the structure of G in the m-neighborhood of v for some $m = m(\epsilon, \lambda_0)$. In other words, for two weighted graphs $G_1 = (V_1, E_1; A_1)$ and $G_2 = (V_2, E_2; A_2)$ and for two vertices $v_1 \in V_1$ and $v_2 \in V_2$ we have

$$\left| \frac{P_{G_1 - v_1}(t)}{P_{G_1}(t)} - \frac{P_{G_2 - v_2}(t)}{P_{G_2}(t)} \right| \leq \epsilon$$

provided the m-neighborhoods of v_1 of G_1 and of v_2 in G_2 are isomorphic.

One particularly interesting case is when $a_e = 1$ for all $e \in E$, when

$$P_G(t) = \sum_{k=0}^{|V|/2} (\text{the number of } k\text{-matchings in } G\,)\, t^k.$$

Then $\beta = \Delta(G)$, the largest degree of a vertex of G. If λ and $\Delta(G)$ are fixed in advance, we obtain a polynomial time algorithm for approximating $x^*_{S,v}$, because the number of different coordinates (S', w') we need to access while computing the (S, v)-coordinate of the iteration $T^m(x)$ grows roughly as $\Delta(G)^m$, which, by (5.2.3.1) bounded by a polynomial in ϵ^{-1}. By looking at the two consecutive iterations of T, in [B+07] a better rate of convergence is established. It is shown that T^2 in fact, a contraction with a factor of

$$1 - \Omega\left(\frac{1}{\sqrt{t\,\Delta(G)}}\right).$$

This phenomenon of fast convergence is called *correlation decay* because to approximate $x^*_{S,v}$ we do not need to care at all about coordinates (S', w') with $\{S', w'\}$ very different from $\{S, v\}$.

We note that once we approximate $x^*_{S,v}$, we can approximate the value of $P_G(t)$ by telescoping. Namely, we number vertices $v_1, \ldots, v_n$ of G and let

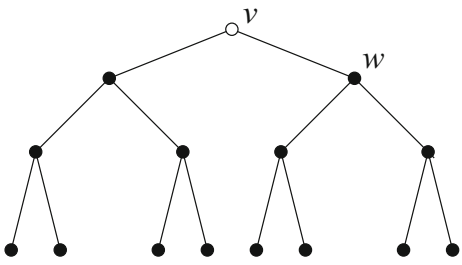
$$\begin{aligned} P_G(t) &= \frac{P_G(t)}{P_{G-v_1}(t)} \frac{P_{G-v_1}(t)}{P_{G-v_1,v_2}(t)} \cdots \frac{P_{G-v_1-\ldots-v_n}(t)}{P_{G-v_1-\ldots-v_{n-1}}(t)} \\ &= \left(x^*_{S_0,v_1} x^*_{S_1,v_2} \cdots x^*_{S_{n-1},v_n}\right)^{-1}, \end{aligned}$$

where $S_0 = \emptyset$, $S_1 = \{v_1\}$, $S_2 = \{v_1, v_2\}, \ldots, S_{n-1} = \{v_1, \ldots, v_{n-1}\}$.

5.2.4 Definition. For positive integers m and $k \geq 2$, we define $\mathbb{T}^k_m$ as the tree with vertices at the levels $0, 1, \ldots, m$, with one vertex, called the *root* at the 0th level connected to $k-1$ vertices at the level 1, and with every vertex at the i-th level connected to one vertex at the $(i-1)$st level and $k-1$ vertices at the $(i+1)$-st level, for $i = 1, \ldots, m-1$. Each vertex at the m-th level is connected to one vertex at the $(m-1)$st level, see Fig. 5.2.

We set the weight on every edge of $\mathbb{T}^k_m$ equal to 1.

Fig. 5.2 The tree $\mathbb{T}^3_3$



5.2.5 Lemma. *Let us fix k and let v^m be the root of $\mathbb{T}^k_m$. For any $t > 0$, we have*

$$\lim_{m\longrightarrow\infty} \frac{P_{\mathbb{T}^k_m - v^m}(t)}{P_{\mathbb{T}^k_m}(t)} = \frac{\sqrt{1+4t(k-1)}-1}{2t(k-1)}.$$

Moreover, for any $t_0 > 0$, the convergence is uniform on the interval $0 < t \leq t_0$.

Proof. As follows from Sect. 5.2.3, for any $\epsilon > 0$ there is $m_0 = m_0(\epsilon, k, t_0)$ such that for $t \leq t_0$ the value of $P_{\mathbb{T}^k_m - v^m}(t)/P_{\mathbb{T}^k_m}(t)$, up to an error ϵ, depends only on the m_0-neighborhood of v^m. However, for all $m \geq m_0$ the m_0-neighborhoods of $v^m \in \mathbb{T}^k_m$ remains the same, from which it follows that the limit in question, call it x, indeed exists.

If we remove the root v_m of $\mathbb{T}^k_m$ and all incident edges, we get a vertex-disjoint union of $(k-1)$ trees $\mathbb{T}^k_{m-1}$, see Fig. 5.2. Hence by (5.2.1.2) the limit x satisfies the equation

$$x = \frac{1}{1+t(k-1)x},$$

from which

$$x = \frac{\sqrt{1+4t(k-1)}-1}{2t(k-1)}.$$

□

We interpret the limit in Lemma 5.2.5 as the limit probability that a random matching in $\mathbb{T}^k_m$ does not contain the root, cf. Sect. 5.2.1.

5.2.6 Regular graphs of large girth. A graph G is called *k-regular* if every vertex of G is incident to precisely k edges. The *girth* of an undirected graph $G = (V, E)$ without loops or multiple edges, denoted $\operatorname{gr} G$, is the smallest number g of vertices of a cycle $v_1 - v_2 - \ldots - v_g - v_1$, where $v_1, \ldots, v_g \in V$ are distinct and $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{g-1}, v_g\}, \{v_g, v_1\} \in E$. If G has no cycles, that is, if G is a forest, we say that $\operatorname{gr} G = +\infty$. Locally (that is, in the vicinity of each vertex), a graph of a large girth looks like a tree, which often allows us to understand the behavior of its matching polynomial.

5.2.7 Lemma. *Let us fix an integer $k > 1$ and let $G_n = (V_n, E_n; \mathbf{1})$, $n \in \mathbb{N}$, be a sequence of k-regular graphs such that $\operatorname{gr} G_n \longrightarrow +\infty$ as $n \longrightarrow \infty$ and with uniform weights equal 1 on every edge of G_n. Let $v_n \in V_n$ be a sequence of vertices. Then, for the matching polynomials $P_{G_n}(t)$ and $P_{G_n - v_n}(t)$ we have*

$$\lim_{n\longrightarrow\infty} \frac{P_{G_n - v_n}(t)}{P_{G_n}(t)} = \frac{2k-2}{k\sqrt{1+4t(k-1)}+k-2} \quad \text{for all} \quad t > 0$$

and the convergence is uniform over $v_n \in V_n$. Moreover, for any fixed $t_0 > 0$, the convergence is also uniform over all $0 \leq t \leq t_0$.

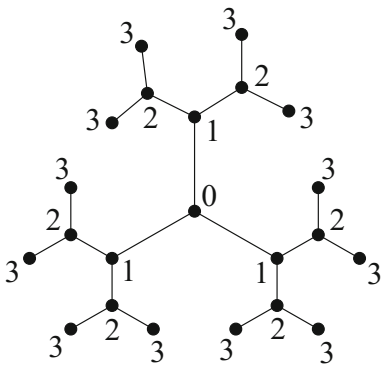


Fig. 5.3 A 3-regular tree with root at 0 and 3 levels

Proof. Let us fix $t_0 > 0$ and an $\epsilon > 0$. As is discussed in Sect. 5.2.3, there exists $m_0 = m_0(\epsilon, t_0, k)$ such that up to an error ϵ, the ratio $P_{G_n - v_n}(t)/P_{G_n}(t)$ depends only on the m_0-neighborhood of v_n in G_n. If $\operatorname{gr} G_n > m$, the neighborhood of v_n looks like the k-regular tree with root at v_n, uniform weight 1 on every edge and m levels, see Fig. 5.3. Hence it follows that the limit in question, say y, indeed exists.

If we remove the vertex v_n with incident edges, then the m-neighborhood of v_n in the resulting graph will be a vertex-disjoint union of k trees $\mathbb{T}^k_{m-1}$. From (5.2.1.2) it follows that

$$y = \frac{1}{1 + tkx},$$

where x is the limit in Lemma 5.2.5. □

Again, we interpret the limit in Lemma 5.2.7 as the limit probability that a random matching in G_n does not contain a particular vertex v_n.

Finally, we compute the logarithmic asymptotic of the partition function $P_{G_n}(t)$ for k-regular graphs of growing girth.

5.2.8 Theorem. *Let us fix an integer $k > 1$ and let $G_n = (V_n, E_n; \mathbf{1})$, $n \in \mathbb{N}$, be a sequence of k-regular graphs such that $\operatorname{gr} G_n \longrightarrow +\infty$ as $n \longrightarrow \infty$ and with uniform weights equal 1 on every edge of G_n. Then, for any $t > 0$ we have*

$$\lim_{n \longrightarrow \infty} \frac{\ln P_{G_n}(t)}{|V_n|} = \frac{k-1}{2} \ln\left(\frac{1 + \sqrt{1 + 4tk - 4t}}{2}\right) - \frac{k-2}{2} \ln\left(\frac{k\sqrt{1 + 4tk - 4t} + k - 2}{2k - 2}\right) + \frac{1}{2} \ln\left(\frac{2kt - 2t}{\sqrt{1 + 4tk - 4t} - 1}\right).$$

Proof. By Lemma 5.2.7 and (5.2.1.3), we have

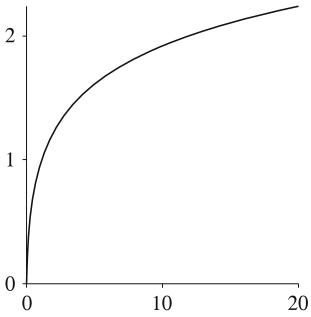


Fig. 5.4 The graph of the limit in Theorem 5.2.8 for 10-regular graphs G_n of growing girth, as a function of t

$$\lim_{n\longrightarrow\infty}\frac{1}{|V_n|}t\frac{d}{dt}\ln P_{G_n}(t)=\frac{1}{2}-\frac{k-1}{k\sqrt{1+4t(k-1)}+k-2},$$

where for any $t_0>0$, the convergence is uniform over $0\le t\le t_0$. Let us fix an $0<\epsilon<t$. Then

$$\lim_{n\longrightarrow\infty}\left(\frac{1}{|V_n|}\ln P_{G_n}(t)-\frac{1}{|V_n|}\ln P_{G_n}(\epsilon)\right)$$
$$=\int_\epsilon^t\left(\frac{1}{2\tau}-\frac{k-1}{k\tau\sqrt{1+4\tau(k-1)}+\tau(k-2)}\right)d\tau. \qquad (5.2.8.1)$$

Since G_n is k-regular, the number of edges of G_n is $k|V_n|/2$ and we can bound

$$\frac{1}{|V_n|}\ln P_{G_n}(\epsilon)\ \le\ \frac{1}{|V_n|}\ln(1+\epsilon)^{k|V_n|/2}\ \le\ \frac{k}{2}\ln(1+\epsilon).$$

One can show that the integrand in (5.2.8.1) is regular at $\tau=0$, and in fact,

$$\frac{1}{2\tau}-\frac{k-1}{k\tau\sqrt{1+4\tau(k-1)}+\tau(k-2)}=\frac{k}{2}+O(\tau)\quad\text{as}\quad\tau\longrightarrow 0+.$$

Hence we can take the limit in (5.2.8.1) as $\epsilon\longrightarrow 0+$. Computing the integral, we complete the proof. □

The graph of the limit for 10-regular graphs as a function of t is pictured on Fig. 5.4.

5.3 Matching Polynomials of Bipartite Graphs

5.3.1 Definition. We consider the special case of $P_G(t)$ for a bipartite graph G. Alternatively, for a given $n\times n$ non-negative matrix $A=(a_{ij})$ and integer $1\le k\le n$, we define

$$\mathrm{per}_k(A) = \sum_{\substack{1\le i_1<\ldots<i_k\le n \\ j_1,\ldots,j_k \text{ pairwise distinct}}} a_{i_1 j_1}\cdots a_{i_k j_k},$$

the sum of permanents of all $k \times k$ submatrices of A and let

$$r_A(t) = \sum_{k=0}^{n} \mathrm{per}_k(A) t^k, \tag{5.3.1.1}$$

where we agree that $\mathrm{per}_0(A) = 1$.

Our exposition loosely follows Csikvári [Cs14] and Lelarge [Le15].

5.3.2 A 2-lift of a graph and a 2-lift of a matrix. Let $G = (V, E; A)$ be an undirected weighted graph without loops or multiple edges. We construct its 2-*lift* H as follows. For each vertex v of G, we introduce two vertices, say v_1 and v_2 of H. For each edge $\{u, v\}$ if G we introduce two edges: either a pair $\{v_1, u_1\}$ and $\{v_2, u_2\}$ of edges or a pair $\{v_1, u_2\}$ and $\{v_2, u_1\}$ of edges (we have a choice here), see Fig. 5.5. We make H a weighted graph by copying the weight of edge e on the lifts of e.

For example, if G is a cycle with n vertices then a 2-lift H can be a pair of vertex disjoint n-cycles of a $2n$-cycle, see Fig. 5.6.

One can similarly define n-lifts. Random lifts of graphs were studied in connection with expander constructions [AL06], but also in connection with perfect matchings [LR05].

Following [Le15], we define a 2-lift of an $n \times n$ matrix $A = \left(a_{ij}\right)$ as a $2n \times 2n$ matrix $B = \left(b_{ij}\right)$, where for all $1 \le i, j \le n$ we have

$$\begin{aligned} \text{either} \quad & b_{ij} = b_{(i+n)(j+n)} = a_{ij} \quad \text{and} \quad b_{(i+n)j} = b_{i(j+n)} = 0 \\ \text{or} \quad & b_{i(j+n)} = b_{(i+n)j} = a_{ij} \quad \text{and} \quad b_{ij} = b_{(i+n)(j+n)} = 0. \end{aligned}$$

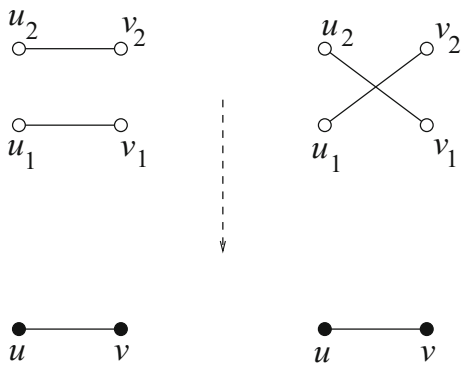
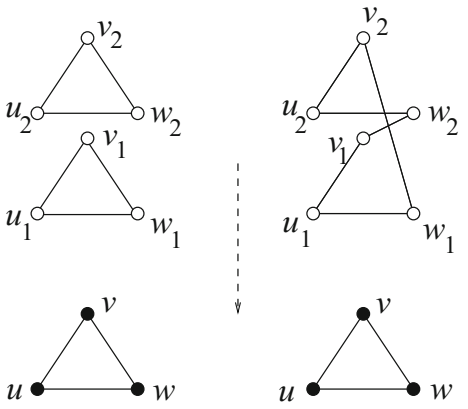


Fig. 5.5 Two ways to lift an edge

Fig. 5.6 2-lifts of a triangle



For example, if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \tag{5.3.2.1}$$

then

$$B = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 4 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 3 & 0 & 0 & 4 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 3 & 4 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \tag{5.3.2.2}$$

are 2-lifts of A. It is clear that our definitions of a 2-lift of a matrix and a 2-lift of a weighted bipartite graph agree.

The following result was proved by Csikvári [Cs14] in the case of uniformly weighted graphs and then extended by Lelarge [Le15] to arbitrary positive weights.

5.3.3 Theorem. *Let G be a weighted bipartite graph with positive weights on the edges and let H be a 2-lift of G. Then*

$$P_H(t) \ \leq \ P_G^2(t) \quad \textit{for all} \quad t \geq 0.$$

Equivalently, if A is an $n \times n$ non-negative matrix and B is a 2-lift of A then for the polynomials $r_A(t)$ and $r_B(t)$ defined by (5.3.1.1), we have

$$r_B(t) \ \leq \ r_A^2(t) \quad \textit{for all} \quad t \geq 0.$$

Proof. Let $\widehat{G}$ be a trivial 2-lift of G consisting of two vertex-disjoint copies of G, say, G_0 and G_1. Since every matching in $\widehat{G}$ can be written uniquely as a disjoint union of a matching in G_0 and a matching in G_1, we deduce that

$$P_{\widehat{G}}(t) = P_G^2(t)$$

(here we don't use that G is bipartite).

Next, we are going to prove that

$$P_H(t) \leq P_{\widehat{G}}(t) \quad \text{for all} \quad t \geq 0. \tag{5.3.3.1}$$

Let $e_1, \ldots, e_k$ be a matching in H and let us consider the edges $f_1, \ldots, f_k$ of G that are the images of $e_1, \ldots, e_k$ under the natural projection $H \longrightarrow G$. Since $e_1, \ldots, e_k$ is a matching, each vertex in G belongs to at most two of the edges $f_1, \ldots, f_k$. Consequently, the multiset $F = \{f_1, \ldots, f_k\}$ is a vertex-disjoint union of edges of mutiplicity 2, paths, and cycles. Since G is a bipartite graph, the cycles have necessarily an even number of vertices.

Let us fix a multiset $F = \{f_1, \ldots, f_k\}$ of edges as above, obtained by a projection of a k-matching in H, and let us compare the total weights $W_H(F)$ and $W_{\widehat{G}}(F)$ of matchings in H and $\widehat{G}$ respectively, projected onto F. If F can be represented as a vertex-disjoint union $F = F_1 \cup F_2$ then clearly

$$W_H(F_1 \cup F_2) = W_H(F_1)W_H(F_2) \quad \text{and} \quad W_{\widehat{G}}(F_1 \cup F_2) = W_{\widehat{G}}(F_1)W_{\widehat{G}}(F_2),$$

so is suffices to compare $W_H(F)$ and $W_{\widehat{G}}(F)$ when F is connected.

If F consists of a single edge of multiplicity 2, then $W_H(F) = W_{\widehat{G}}(F)$, as there are exactly two edges of equal weight projected onto the edge in F, see Fig. 5.5.

If F is a path then $W_H(F) = W_{\widehat{G}}(F)$, since if F is a projection of a matching in H, there are exactly two matchings in H of the same weight projecting into F, whose union consists of two vertex-disjoint paths projected onto F. Similarly, there are exactly two matchings in $\widehat{G}$ projected onto F whose union consists of two vertex-disjoint paths projected onto F, see Fig. 5.7.

In particular, every path in G can be lifted to two matchings in $\widehat{G}$.

Finally, if F is an even cycle then $W_H(F) = W_{\widehat{G}}(F)$. If a matching in H is projected onto an even cycle in G, then there are exactly two such matchings in H. Similarly, there are two vertex-disjoint cycles in $\widehat{G}$ projected onto F containing two matchings in $\widehat{G}$ projected onto F, see Fig. 5.8.

This concludes the proof of (5.3.3.1) and hence the proof of the theorem. □

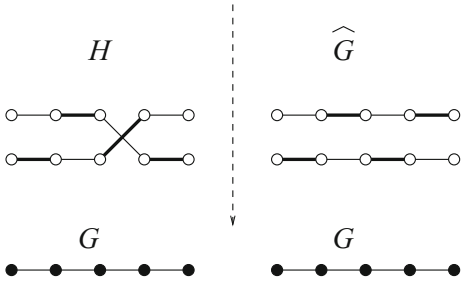


Fig. 5.7 If there is a matching in H projecting onto a path in G then there are exactly two such matchings in H (one of *thick lines* and the other of *thin lines*). Similarly, there are exactly two matchings in $\widehat{G}$ projecting onto the same path in G

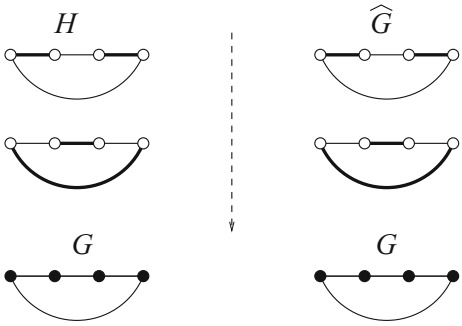


Fig. 5.8 If there is a matching in H projecting onto an even cycle, then there are exactly two such matchings (one of *thick lines* and the other of *thin lines*). Similarly, there are exactly two matchings in $\widehat{G}$ projected onto the cycle

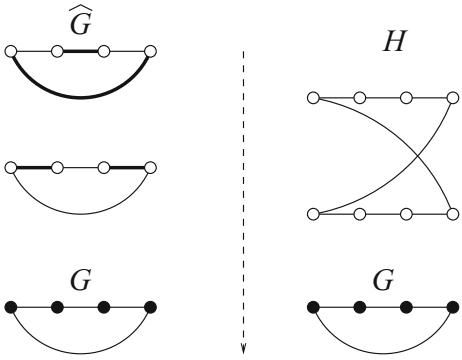


Fig. 5.9 If a 4-cycle in G is lifted to two vertex-disjoint 4-cycles in H then there is a 4-matching in H projected onto the cycle. If a 4-cycle in G is lifted to an 8-cycle in H then there is no 4-matching in H projected onto the cycle

Some remarks are in order. We may have $P_H(t) < P_G^2(t) = P_{\widehat{G}}(t)$, since there can be a $2k$-cycle in G which is the projection of a $4k$-cycle in H. In that case, there is a $2k$-matching in $\widehat{G}$ projecting onto the $2k$-cycle but there is no $2k$-matching in H projecting onto that $2k$-cycle, see Fig. 5.9.

For example, for matrix A of (5.3.2.1), we have per $B =$ per $C = 100 = 10^2$ for 2-lifts B and C of (5.3.2.2), while per $D = 52 < 10^2$ for the 2-lift D of (5.3.2.2).

We note that if G is a triangle and H is a 2-lift that is a 6-cycle such as on Fig. 5.6, then $P_G(t)$ is a polynomial of degree 1 (since the maximum matching in G consists of one edge) while $P_H(t)$ is a polynomial of degree 3 (since the maximum matching in H consists of 3 edges). Therefore, Theorem 5.3.3 does not hold if G is not required to be bipartite. For non-bipartite graphs, the proof breaks down at the last step: if F is an odd cycle which is a projection of a matching in H, then F is a projection of exactly two such matchings whose union is an even cycle of twice the length of F and not two vertex-disjoint copies of F, see Fig. 5.10.

Applying 2-lifts repeatedly, one can obtain from a general graph a graph with a larger girth that locally looks more and more like a tree.

The following result and its proof is attributed to Linial in [Cs14, Le15].

5.3.4 Lemma. *Let $G = (V, E)$ be an undirected graph without loops or multiple edges. Then there is a graph H obtained by repeated applications of 2-lifts to G such that* gr $H >$ gr G.

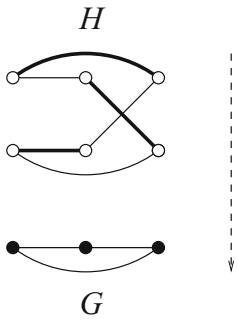


Fig. 5.10 If there is a matching projecting onto a cycle of length 3, then there are exactly two such matchings (one of *thick lines* and the other of *thin lines*) whose union is an even cycle of length 6

Proof. Suppose that gr $G = g$ and let k be the number of cycles of length g. Let $\widehat{G}$ be a *random* 2-lift of G, where independently for each edge $\{u, v\}$ of G, we choose the lift $\{u_1, v_1\}$ and $\{u_2, v_2\}$ or the lift $\{u_1, v_2\}$ and $\{u_2, v_1\}$, with probability $1/2$ each, see Sect. 5.3.2. Then a g-cycle in G is lifted to a pair of g-cycles in $\widehat{G}$ with probability $1/2$ and to a $2g$-cycle in $\widehat{G}$ with probability $1/2$. Indeed, a path $v_1 - v_2 - \ldots - v_g$ of length g is lifted to a pair of paths of length g each in $\widehat{G}$, and then the closing edge $v_1 - v_g$ is either lifted to a pair of edges closing each path to a cycle of length g or to a pair of edges patching the paths into a cycle of length $2g$, see Fig. 5.9. Consequently, for every 2-lift $\widehat{G}$ of G we have gr $\widehat{G} \geq g$ and the expected number of g-cycles in $\widehat{G}$ is k. Since with positive probability $\widehat{G}$ consists of two vertex-disjoint copies of G, in which case the number of g-cycles in $\widehat{G}$ is $2k > k$, there is a lift $\widehat{G}$ which has fewer than k cycles of length g. Iterating, we conclude that there is a sequence of 2-lifts of G which produces a graph H with no g-cycles, in which case gr $H >$ gr G. □

As a corollary of Lemma 5.3.4, Theorems 5.3.3 and 5.2.8, we obtain the following lower bound.

5.3.5 Theorem. *Let $G = (V, E; \mathbf{1})$ be a k-regular bipartite graph with uniform weights 1 on all edges. Then, for $k \geq 2$,*

$$\begin{aligned}\frac{\ln P_G(t)}{|V|} \geq\ & \frac{k-1}{2} \ln\left(\frac{1+\sqrt{1+4tk-4t}}{2}\right) \\ & - \frac{k-2}{2} \ln\left(\frac{k\sqrt{1+4tk-4t}+k-2}{2k-2}\right) \\ & + \frac{1}{2} \ln\left(\frac{2kt-2t}{\sqrt{1+4tk-4t}-1}\right).\end{aligned}$$

for all $t > 0$.

Proof. Using Lemma 5.3.4, for $n = 1, \ldots,$ we construct an infinite sequence $G_n = (V_n, E_n; \mathbf{1})$ of graphs where $G_1 = G$, graph G_{n+1} is a 2-lift of G_n for all n and gr $G_n \longrightarrow +\infty$ as $n \longrightarrow \infty$. Since $|V_{n+1}| = 2|V_n|$, from Theorem 5.3.3 we conclude that

$$\frac{\ln P_{G_n}(t)}{|V_n|} \quad \text{for} \quad n = 1, \ldots,$$

is a non-increasing sequence. The proof now follows from Theorem 5.2.8. □

Taking the limit as $t \longrightarrow +\infty$, we obtain a lower bound for the number of perfect matchings in a k-regular bipartite graph.

5.3.6 Theorem. *Let A be an $n \times n$ matrix with 0-1 entries such that every row and every column of A contains exactly k 1s. Then*

$$\frac{\ln \operatorname{per} A}{n} \geq (k-1)\ln(k-1) - (k-2)\ln k.$$

Moreover, there is a sequence of $\{A_n\}$ of $n \times n$ matrices with 0-1 entries, each containing exactly k 1s such that

$$\lim_{n\longrightarrow\infty} \frac{\ln \operatorname{per} A_n}{n} = (k-1)\ln(k-1) - (k-2)\ln k.$$

Proof. Let $r_A(t)$ be the matching polynomial of A, see Definition 5.3.1. Then $r_A(t)$ is a polynomial of degree n with the coefficient of t^n equal to per $A > 0$, see, for example, Theorem 3.3.2.

Therefore,

$$\lim_{t\longrightarrow+\infty} \left(\frac{\ln r_A(t)}{n} - \ln t \right) = \frac{\ln \operatorname{per} A}{n}.$$

On the other hand, by Theorem 5.3.5,

$$\begin{aligned} \frac{\ln r_A(t)}{n} \geq\ & (k-1)\ln\left(\frac{1+\sqrt{1+4tk-4t}}{2}\right) \\ & - (k-2)\ln\left(\frac{k\sqrt{1+4tk-4t}+k-2}{2k-2}\right) \\ & + \ln\left(\frac{2kt-2t}{\sqrt{1+4tk-4t}-1}\right) \end{aligned}$$

and hence

$$\frac{\ln r_A(t)}{n} - \ln t \geq (k-1)\ln\left(\frac{1+\sqrt{1+4tk-4t}}{2\sqrt{t}}\right) - (k-2)\ln\left(\frac{k\sqrt{1+4tk-4t}+k-2}{(2k-2)\sqrt{t}}\right) + \ln\left(\frac{2kt-2t}{\left(\sqrt{1+4tk-4t}-1\right)\sqrt{t}}\right).$$

Taking the limit as $t \longrightarrow +\infty$, we obtain

$$\begin{aligned}\frac{\ln \operatorname{per} A}{n} &\geq \frac{k-1}{2}\ln(k-1) - \frac{k-2}{2}\ln\frac{k^2}{k-1} + \frac{1}{2}\ln(k-1)\\ &= (k-1)\ln(k-1) - (k-2)\ln k,\end{aligned}$$

as required.

As in the proof of Theorem 5.3.5, matrices $\{A_n\}$ are obtained as adjacency matrices of graphs G_n that are repeated 2-lifts of a given bipartite k-regular graph and such that gr $G_n \longrightarrow +\infty$ as $n \longrightarrow \infty$. □

We can rewrite the bound of Theorem 5.3.6 as

$$\operatorname{per} A \geq k^n \left(\frac{k-1}{k}\right)^{(k-1)n},$$

in which case it becomes the familiar bound (3.3.5.1).

5.3.7 Upper bounds for the matching polynomial of a k-regular graph. In [D+15], Davies, Jenssen, Perkins and Roberts prove that if $G = (V, E)$ is a k-regular graph then, for any $t > 0$, the quantity

$$\frac{t}{|E|}\frac{d}{dt}\ln P_G(\lambda) = \frac{t}{|E|}\frac{P'_G(t)}{P_G(t)}$$

attains its maximum when G is the k-regular complete bipartite graph, see Fig. 5.11. This quantity is naturally interpreted as the expected proportion of the edges of G covered by random matching in G, where the probability that a random matching contains exactly s edges is proportional to t^s.

As is remarked in [D+15], this implies that for any $t > 0$, the maximum of

$$P_G^{1/|V|}(t)$$

over k-regular graphs $G = (V, E)$ is attained when G is the complete bipartite graph.

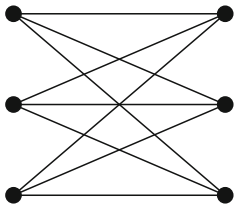


Fig. 5.11 The complete bipartite 3-regular graph

5.4 The Bethe-Entropy Lower Bound

The goal of this section is to prove the following result due to Lelarge [Le15].

5.4.1 Theorem. *For a positive $n \times n$ matrix $A = \left(a_{ij}\right)$ and a real $t > 0$, let*

$$r_A(t) = \sum_{k=0}^{n} \mathrm{per}_k(A) t^k$$

be the matching polynomial, where $\mathrm{per}_k(A)$ is the sum of the permanents of all $k \times k$ submatrices of A and $\mathrm{per}_0(A) = 1$.

On the set $\mathcal{M}_n$ of $n \times n$ nonnegative real matrices $X = \left(x_{ij}\right)$ such that

$$\sum_{j=1}^{n} x_{ij} \le 1 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} x_{ij} \le 1 \quad \text{for} \quad j = 1, \ldots, n$$

let us define a function

$$\begin{aligned} f_{A,t}(X) = \sum_{i,j=1}^{n} \ln\left(t a_{ij}\right) x_{ij} - \sum_{i,j=1}^{n} x_{ij} \ln x_{ij} + \sum_{i,j=1}^{n} (1 - x_{ij}) \ln(1 - x_{ij}) \\ - \sum_{i=1}^{n} \left(1 - \sum_{j=1}^{n} x_{ij}\right) \ln\left(1 - \sum_{j=1}^{n} x_{ij}\right) \\ - \sum_{j=1}^{n} \left(1 - \sum_{i=1}^{n} x_{ij}\right) \ln\left(1 - \sum_{i=1}^{n} x_{ij}\right). \end{aligned}$$

Then $f_{A,t}$ is strictly concave on $\mathcal{M}_n$, attains maximum on $\mathcal{M}_n$ at a unique point and

$$\ln r_A(t) \ge \max_{X \in \mathcal{M}_n} f_{A,t}(X).$$

Taking the limit as $t \longrightarrow +\infty$, we obtain a lower bound for the permanent.

5.4.2 Theorem. *Let $\mathcal{B}_n$ be the polytope of $n \times n$ doubly stochastic matrices, that is, non-negative matrices $X = \left(x_{ij}\right)$ such that*

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \textit{for} \quad i = 1, \ldots, n \quad \textit{and} \quad \sum_{i=1}^{n} x_{ij} = 1 \quad \textit{for} \quad j = 1, \ldots, n.$$

For a positive $n \times n$ matrix $A = \left(a_{ij}\right)$ and $X \in \mathcal{B}_n$, let

$$g_A(X) = \sum_{i,j=1}^{n} x_{ij} \ln \frac{a_{ij}}{x_{ij}} + \sum_{i,j=1}^{n} \left(1 - x_{ij}\right) \ln \left(1 - x_{ij}\right).$$

Then g_A is a concave function and

$$\ln \operatorname{per} A \;\geq\; \max_{X \in \mathcal{B}_n} g_A(X).$$

The inequality of Theorem 5.4.2 was conjectured by Vontobel [Vo13] and deduced by Gurvits [Gu11] from Schrijver's inequality [Sc98]. We take a different route here, due to Lelarge [Le15], first proving Theorem 5.4.1 and then obtaining Theorem 5.4.2 as a limit case. If $A = \left(a_{ij}\right)$ is a doubly stochastic matrix, from Theorem 5.4.2 we get

$$\ln \operatorname{per} A \;\geq\; g_A(A) = \sum_{i,j=1}^{n} \left(1 - a_{ij}\right) \ln \left(1 - a_{ij}\right),$$

which is Schrijver's inequality.

The lower bounds for $\ln r_A(t)$ of Theorem 5.4.1 and for $\ln \operatorname{per} A$ of Theorem 5.4.2 are known as the *Bethe-entropy lower bounds*. Their advantage is that they supply an easily computable lower bound as a solution to a convex optimization problem.

We prove Theorem 5.4.1 by taking a closer look at the lift of an arbitrary positive matrix. We follow Lelarge [Le15] with some modifications.

Let $A = \left(a_{ij}\right)$ be an $n \times n$ positive matrix, which we interpret as the matrix of weights on the complete bipartite graph $K_{n,n}$ with vertices $1L, \ldots, nL$ and $1R, \ldots, nR$, so that the weight on the edge iL and jR is a_{ij}, cf. Sect. 3.1.2. As we iterate 2-lifts m times, as described in Sect. 5.3.2, we obtain a graph G with $N = 2^{m+1}n$ vertices, where each vertex has *type* $1L, \ldots, nL$ or $1R, \ldots, nR$, depending on where it projects under the natural projection $G \longrightarrow K_{n,n}$. Each vertex of type iL is connected by an edge to a vertex of type jR with weight a_{ij} on the edge for $j = 1, \ldots, n$ and each vertex of type jR is connected by an edge to a vertex of type iL with weight a_{ij} on the edge for $i = 1, \ldots, n$, see Fig. 5.12.

In particular, $G = (V, E)$ is an n-regular graph. Our goal is to compute the asymptotic of $\ln P_G(t)/|V|$ as the girth of G grows. First, we prove a refinement of Lemma 5.2.5, for which we introduce trees $\mathcal{L}_m^{ij}$ and $\mathcal{R}_m^{ij}$ that are refinements of trees $\mathbb{T}_m^k$ from Sect. 5.2.4.

5.4.3 Definition. The tree $\mathcal{L}_m^{ij}$ is a tree with m levels that has the root of type iL at the level 0 connected to $n - 1$ vertices at the level 1 of type kR for all $k \neq j$. Every vertex at the level 1 is connected to n vertices of the type kL for $k = 1, \ldots, n$, one

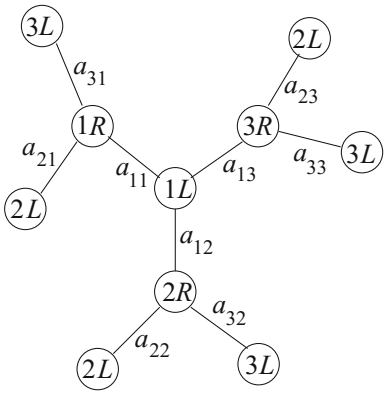


Fig. 5.12 A part of the lift of a 3×3 matrix

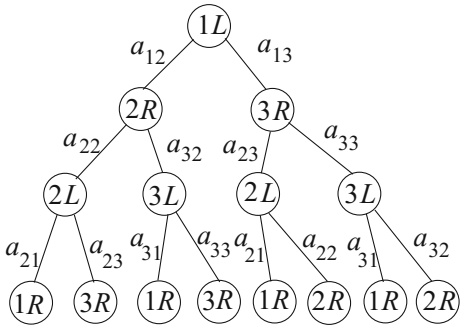


Fig. 5.13 The $\mathcal{L}_3^{11}$ tree for $n = 3$

of which is the root while the other $n - 1$ are at the level 2. Every vertex at the level 2 is connected to n vertices of the type kR for $k = 1, \ldots, n$, one of which is at the level 1 and the other $n - 1$ are at the level 3, etc.

The tree $\mathcal{R}_m^{ij}$ is a tree with m levels that has the root of type jR at the level 0 connected to $n - 1$ vertices at the level 1 of type kL for all $k \neq i$. Every vertex at the level 1 is connected to n vertices of the type kR for $k = 1, \ldots, n$, one of which is the root while the other $n - 1$ are at the level 2. Every vertex at the level 2 is connected to n vertices of the type kL for $k = 1, \ldots, n$, one of which is at the level 1 and the other $n - 1$ are at the level 3, etc.

If we remove an edge connecting vertices of the type iL and jR in a lift of a matrix, in the neighborhood of the removed edge, the lift will look like the union of two trees of the types $\mathcal{L}^{ij}$ and $\mathcal{R}^{ij}$, see Figs. 5.12 and 5.13.

The weights on the edges are replicated in the usual way: an edge connecting vertices of types iL and jR has weight a_{ij}, see Fig. 5.13.

5.4.4 Lemma. *Let us fix a positive $n \times n$ matrix $A = (a_{ij})$. Let v_m denote the root of $\mathcal{L}_m^{ij}$, respectively $\mathcal{R}_m^{ij}$. For every $t > 0$ the limits*

$$\lim_{m\longrightarrow\infty}\frac{P_{\mathcal{L}_m^{ij}-v_m}(t)}{P_{\mathcal{L}_m^{ij}}(t)}=l_{ij}=l_{ij}(A,t)$$

and

$$\lim_{m\longrightarrow\infty}\frac{P_{\mathcal{R}_m^{ij}-v_m}(t)}{P_{\mathcal{R}_m^{ij}}(t)}=r_{ij}=r_{ij}(A,t)$$

exist and satisfy the system of equations

$$l_{ij}=\left(1+t\sum_{k:\,k\neq j}a_{ik}r_{ik}\right)^{-1}\quad\text{and}\quad r_{ij}=\left(1+t\sum_{k:\,k\neq i}a_{kj}l_{kj}\right)^{-1}$$

for all $1\leq i,j\leq n$. *Moreover, for any* $t_0>0$ *the convergence is uniform over all* $0<t\leq t_0$.

Proof. The proof is a refinement of that of Lemma 5.2.5. As follows from Sect. 5.2.3, for any $\epsilon>0$ there is $m_0=m_0(\epsilon,A,t_0)$ such that for $t<t_0$, the value of $P_{\mathcal{L}_m^{ij}-v_m}(t)/P_{\mathcal{L}_m^{ij}}(t)$, up to an error ϵ, depends only on the m_0-neighborhood of v_m in $\mathcal{L}_m^{ij}$. However, for $m>m_0$ the m_0 neighborhood of v_m in $\mathcal{L}_m^{ij}$ remains the same, from which it follows that the limit l_{ij} indeed exists. The existence of the limit r_{ij} is proved similarly.

If we remove the root v_m of $\mathcal{L}_m^{ij}$ with all incident edges, we get a vertex-disjoint union of $n-1$ trees $\mathcal{R}_{m-1}^{ik}$ for $k\neq j$ and if we remove the root of v_m of $\mathcal{R}_m^{ij}$ with all incident edges, we get a vertex-disjoint union of $n-1$ trees $\mathcal{L}_{m-1}^{kj}$ for $k\neq i$. The equations for l_{ij} and r_{ij} then follow from (5.2.1.2). □

A crucial observation of Lelarge [Le15] relates the probabilities l_{ij} and r_{ij} to the solution of a convex optimization problem.

5.4.5 Lemma. *Let us fix an* $n\times n$ *positive matrix* $A=\left(a_{ij}\right)$ *and* $t>0$. *Let the set* $\mathcal{M}_n$ *of matrices* $X=\left(x_{ij}\right)$ *and a function* $f=f_{A,t}$ *be defined as in Theorem 5.4.1. Then* f *is a strictly concave function.*

Let $l_{ij}=l_{ij}(A,t)$ *and* $r_{ij}=r_{ij}(A,t)$ *be the probabilities from Lemma 5.4.4. Then the matrix* $X^*=X^*(A,t)=\left(x_{ij}^*\right)$ *defined by*

$$x_{ij}^*=\frac{ta_{ij}l_{ij}r_{ij}}{1+ta_{ij}l_{ij}r_{ij}}$$

is the maximum point of f *on* $\mathcal{M}_n$.

Proof. For $i=1,\ldots,n$, let us define

$$g_i(X) = -\sum_{j=1}^{n} x_{ij} \ln x_{ij} + \sum_{j=1}^{n} (1 - x_{ij}) \ln(1 - x_{ij}) - \left(1 - \sum_{j=1}^{n} x_{ij}\right) \ln \left(1 - \sum_{j=1}^{n} x_{ij}\right) + \left(\sum_{j=1}^{n} x_{ij}\right) \ln \left(\sum_{j=1}^{n} x_{ij}\right)$$

and

$$u_i(X) = \left(\sum_{j=1}^{n} x_{ij}\right) \ln \left(\sum_{j=1}^{n} x_{ij}\right).$$

For $j = 1, \ldots, n$, let us similarly define

$$h_j(X) = -\sum_{i=1}^{n} x_{ij} \ln x_{ij} + \sum_{i=1}^{n} (1 - x_{ij}) \ln(1 - x_{ij}) - \left(1 - \sum_{i=1}^{n} x_{ij}\right) \ln \left(1 - \sum_{i=1}^{n} x_{ij}\right) + \left(\sum_{i=1}^{n} x_{ij}\right) \ln \left(\sum_{i=1}^{n} x_{ij}\right)$$

and

$$v_j(X) = \left(\sum_{i=1}^{n} x_{ij}\right) \ln \left(\sum_{i=1}^{n} x_{ij}\right).$$

From Sect. 2.1.3, the functions $g_i(X)$ and $h_j(X)$ are concave, while from Sect. 2.1.1.2, the functions u_i and v_j are convex.

Since we have

$$f(X) = \sum_{i,j=1}^{n} \ln \left(t a_{ij}\right) x_{ij} + \sum_{i=1}^{n} g_i(X) + \sum_{j=1}^{n} h_j(X) - \sum_{i=1}^{n} u_i(X) - \sum_{j=1}^{n} v_j(X),$$

we conclude that $f(X)$ is concave. Moreover, since $\sum_{i=1}^{n} u_i(X) + \sum_{j=1}^{n} v_j(X)$ is a strictly convex function, the function $f(X)$ is strictly concave.

To check that X^* is indeed the maximum point of f, we compute the gradient of f.

We have

$$\frac{\partial}{\partial x_{ij}} f(X) = \ln \left(t a_{ij}\right) - \ln x_{ij} - \ln \left(1 - x_{ij}\right) + \ln \left(1 - \sum_{k=1}^{n} x_{ik}\right) + \ln \left(1 - \sum_{k=1}^{n} x_{kj}\right).$$

Using the equations of Lemma 5.4.4, we write

$$t a_{ij} l_{ij} r_{ij} = \frac{t a_{ij} l_{ij}}{1 + t \sum_{k:\, k \neq i} a_{kj} l_{kj}}$$

and

$$1 + ta_{ij}l_{ij}r_{ij} = 1 + \frac{ta_{ij}l_{ij}}{1 + t\sum_{k:\ k\neq i} a_{kj}l_{kj}} = \frac{1 + t\sum_{k=1}^{n} a_{kj}l_{kj}}{1 + t\sum_{k:\ k\neq i} a_{kj}l_{kj}},$$

from which

$$x^*_{ij} = \frac{ta_{ij}l_{ij}}{1 + t\sum_{k=1}^{n} a_{kj}l_{kj}}. \tag{5.4.5.1}$$

Similarly, we write

$$ta_{ij}l_{ij}r_{ij} = \frac{ta_{ij}r_{ij}}{1 + t\sum_{k:\ k\neq j} a_{ik}r_{ik}}$$

and

$$1 + ta_{ij}l_{ij}r_{ij} = \frac{1 + t\sum_{k=1}^{n} a_{ik}r_{ik}}{1 + t\sum_{k:\ k\neq j} a_{ik}r_{ik}}$$

from which

$$x^*_{ij} = \frac{ta_{ij}r_{ij}}{1 + t\sum_{k=1}^{n} a_{ik}r_{ik}}. \tag{5.4.5.2}$$

It follows from (5.4.5.1) that

$$1 - \sum_{k=1}^{n} x^*_{kj} = \left(1 + t\sum_{k=1}^{n} a_{kj}l_{kj}\right)^{-1} \tag{5.4.5.3}$$

and it follows from (5.4.5.2) that

$$1 - \sum_{k=1}^{n} x^*_{ik} = \left(1 + t\sum_{k=1}^{n} a_{ik}r_{ik}\right)^{-1}. \tag{5.4.5.4}$$

In particular, it follows from (5.4.5.3) and (5.4.5.4) that X^* is a feasible point of f.

Using (5.4.5.1)–(5.4.5.4) we obtain that

$$\begin{aligned}\frac{\partial}{\partial x_{ij}} f(X)\Big|_{X=X^*} &= \ln\left(ta_{ij}\right) - \ln\frac{ta_{ij}l_{ij}}{1 + t\sum_{k=1}^{n} a_{kj}l_{kj}} - \ln\frac{1 + t\sum_{k:\ k\neq j} a_{ik}r_{ik}}{1 + t\sum_{k=1}^{n} a_{ik}r_{ik}} \\ &\quad - \ln\left(1 + t\sum_{k=1}^{n} a_{ik}r_{ik}\right) - \ln\left(1 + t\sum_{k=1}^{n} a_{kj}l_{kj}\right) \\ &= -\ln l_{ij} - \ln\left(1 + t\sum_{k:\ k\neq j} a_{ik}r_{ik}\right) = 0.\end{aligned}$$

Hence the gradient of a strictly concave function f at a feasible point X^* is 0 and X^* is the maximum point of f. □

Next, we prove a refinement of Lemma 5.2.7.

5.4.6 Lemma. *Let us fix a positive $n \times n$ matrix $A = (a_{ij})$, a real $t > 0$ and let $X^* = X^*(A, t) = \left(x^*_{ij}\right)$ be the maximum point of the function $f_{A,t}(X)$ in Lemma 5.4.5.*

Let G_m be a sequence of weighted graphs obtained from the complete bipartite graph $K_{n,n}$ with weights A by a repeated application of 2-lifts and such that gr $G_m \longrightarrow +\infty$ *as* $m \longrightarrow \infty$. *Let v_m be a vertex of G_m. Then*

$$\lim_{m\longrightarrow\infty} \frac{P_{G_m - v_m}(t)}{P_{G_m}(t)} = 1 - \sum_{j=1}^{n} x^*_{ij} \quad \textit{provided} \quad v_m \quad \textit{is of type} \quad iL$$

and

$$\lim_{m\longrightarrow\infty} \frac{P_{G_m - v_m}(t)}{P_{G_m}(t)} = 1 - \sum_{i=1}^{n} x^*_{ij} \quad \textit{provided} \quad v_m \quad \textit{is of type} \quad jR.$$

The convergence is uniform over $v_m \in G_m$. Moreover, for any fixed t, the convergence is also uniform over all $0 \le t \le t_0$.

Proof. We begin as in the proof of Lemma 5.2.7. Let us fix $t_0 > 0$ and an $\epsilon > 0$. As is discussed in Sect. 5.2.3, there exists $m_0 = m_0(\epsilon, A, t)$ such that up to an error ϵ, the ratio $P_{G_m - v_m}(t)/P_{G_m}(t)$ depends only on the m_0-neighborhood of v_m. However, if gr $G_m > m_0$, the m_0-neighborhood depends only on the type of the vertex v_m, see Fig. 5.13, from which it follows that the limit indeed exists.

If v_m is of type iL and gr $G_m > s$ then in the s-neighborhood of v_m, the graph $G_m - v_m$ looks like a vertex-disjoint union of n trees $\mathcal{R}^{s-1}_{ij}$, for $j = 1, \ldots, n$. Therefore, by (5.2.1.2) and Lemma 5.4.4, the limit is equal to

$$\left(1 + t\sum_{j=1}^{n} r_{ij}\right)^{-1} = 1 - \sum_{j=1}^{n} x^*_{ij},$$

where the last equation follows by (5.4.5.4).

If v_m is of type jL and gr $G_m > s$ then in the s-neighborhood of v_m, the graph $G_m - v_m$ looks like a vertex-disjoint union of n trees $\mathcal{L}^{s-1}_{ij}$, for $i = 1, \ldots, n$. Therefore, by (5.2.1.2) and Lemma 5.4.4, the limit is equal to

$$\left(1 + t\sum_{i=1}^{n} l_{ij}\right)^{-1} = 1 - \sum_{i=1}^{n} x^*_{ij},$$

where the last equation follows by (5.4.5.3). □

5.4.7 Proof of Theorem 5.4.1. Let us fix a $t_0 > 0$. For $m = 1, 2, \ldots$, let G_m be a weighted graph obtained from $K_{n,n}$ with matrix A of weights by a sequence of m 2-lifts and such that gr $G_m \longrightarrow +\infty$ as $m \longrightarrow \infty$, see Lemma 5.3.4. Then G_m has $n2^{m+1}$ vertices. For each $i = 1, \ldots, n$, exactly 2^m of the vertices have type iL and for each $j = 1, \ldots, n$, exactly 2^m of the vertices have type jR. Applying (5.2.1.3) and Lemma 5.4.6, we conclude that

$$\lim_{m\longrightarrow\infty} \frac{d}{dt} \frac{\ln P_{G_m}(t)}{n2^{m+1}} = \frac{1}{2nt} \sum_{i,j=1}^{n} x^*_{ij}(t), \tag{5.4.7.1}$$

and the convergence is uniform for all $0 < t \leq t_0$.

On the other hand, since the function $t \longmapsto f_{A,t}(X)$ is smooth and strictly concave, the maximum point $X^*(t)$ depends smoothy on t. Since $X^*(t)$ is the maximum point, we get

$$\frac{\partial}{\partial x_{ij}} f_t(X)\Big|_{X=X^*(t)} = 0 \quad \text{for all} \quad i, j$$

and, therefore,

$$\begin{aligned} \frac{d}{dt} f_t\left(X^*(t)\right) &= \sum_{i,j=1}^{n} \left(\frac{\partial}{\partial x_{ij}} f_t(X)\Big|_{X=X^*(t)}\right)\left(\frac{d}{dt} x^*_{ij}(t)\right) + \frac{\partial}{\partial t} f_t(X)\Big|_{X=X^*(t)} \\ &= \frac{1}{t} \sum_{i,j=1}^{n} x^*_{ij}(t). \end{aligned}$$

Therefore, by (5.4.7.1)

$$\lim_{m\longrightarrow\infty} \frac{d}{dt} \frac{\ln P_{G_m}(t)}{n2^{m+1}} = \frac{1}{2n} \frac{d}{dt} f_t\left(X^*(t)\right) \tag{5.4.7.2}$$

and the convergence is uniform over all $0 < t \leq t_0$.

As is easy to see,

$$\lim_{t\longrightarrow 0+} f_t\left(X^*(t)\right) = \lim_{t\longrightarrow 0+} \max_{X\in\mathcal{M}_n} f_t(X) = 0$$

and, as in the proof of Theorem 5.2.8, we have

$$\lim_{t\longrightarrow 0+} \frac{\ln P_{G_m}(t)}{n2^{m+1}} = 0.$$

Then from (5.4.7.2), we obtain

$$\lim_{m\longrightarrow\infty} \frac{\ln P_{G_m}(t)}{n2^{m+1}} = \frac{1}{2n} f_t\left(X^*(t)\right) = \frac{1}{2n} \max_{X\in\mathcal{M}_n} f_t(X) \tag{5.4.7.3}$$

for all $0 < t \leq t_0$ and hence t_0 was chosen arbitrarily, (5.4.7.3) holds for all $t > 0$. Since by Theorem 5.3.3, we have

$$\frac{\ln r_A(t)}{2n} \leq \frac{\ln P_{G_m}(t)}{n2^{m+1}},$$

the proof follows. □

5.4.8 Proof of Theorem 5.4.2. Since $r_A(t)$ is a polynomial with the highest term (per A) t^n with per $A > 0$, we have

$$\lim_{t \longrightarrow +\infty} \ln r_A(t) - n \ln t = \ln \text{per } A. \tag{5.4.8.1}$$

By Sect. 2.1.3, $g_A(X)$ is a continuous concave function and hence the maximum of g_A on $\mathcal{B}_n$ is attained, say, at doubly stochastic matrix X^*. Then $X^* \in \mathcal{M}_n$, where $\mathcal{M}_n$ is the set of matrices defined in Theorem 5.4.1 and hence by Theorem 5.4.1, for every $t > 0$ we have

$$\ln r_A(t) - n \ln t \geq f_{A,t}\left(X^*\right) - n \ln t = g_A(X^*) - n \ln t + (\ln t) \sum_{i,j=1}^{n} x_{ij}^* = g_A\left(X^*\right)$$

and the proof follows by (5.4.8.1).

□

5.5 Hypergraph Matching Polynomial

5.5.1 The matching polynomial of a hypergraph. Let $H = (V, E)$ be a d-uniform hypergraph with set V of vertices and set E of edges. Hence each edge $e \in E$ is a set of d vertices from V. A *matching* in H is a set of pairwise vertex-disjoint edges. Given complex weights $w : E \longrightarrow \mathbb{C}$ on the edges of H, we define the *weight* of a matching $\{e_1, \ldots, e_k\}$ by $w(e_1) \cdots w(e_k)$. We consider the matching with no edges as having weight 1. We define the matching polynomial of H by

$$P_H(w) = \sum_{\substack{e_1,\ldots,e_k \in E: \\ e_1,\ldots,e_k \text{ is a matching}}} w(e_1) \cdots w(e_k),$$

where the sum includes all matchings in H, including the empty matching with the corresponding product equal to 1.

The following result bounds from below the distance from complex zeros of $P_H(w)$ to the origin.

5.5.2 Theorem. *Let $H = (V, E)$ be a d-uniform hypergraph for $d > 1$ and let $w : E \longrightarrow \mathbb{C}$ be complex weights such that*

$$\sum_{\substack{e \in E: \\ v \in e}} |w(e)| \leq \frac{(d-1)^{d-1}}{d^d} \quad \text{for all} \quad v \in V.$$

Then

$$P_H(w) \neq 0.$$

Proof. Given a set $S \subset V$ of vertices, let $H - S$ be the hypergraph with set $V \setminus S$ of vertices and consisting of the edges of H that do not intersect S. We denote the restriction of $w : E \longrightarrow \mathbb{C}$ onto $H - S$ also by w. Then for any vertex $v \in V$, we have

$$P_H(w) = P_{H-v}(w) + \sum_{\substack{e \in E: \\ v \in e}} w(e) P_{H-e}(w), \tag{5.5.2.1}$$

where $P_{H-v}(w)$ accounts for all matchings not containing v, whereas the sum accounts for all matchings containing v (we use $H - v$ as a shorthand for $H - \{v\}$).

We prove by induction on $|V|$ that $P_H(w) \neq 0$ and, moreover, for any vertex v of V, we have

$$\left| 1 - \frac{P_{H-v}(w)}{P_H(w)} \right| \leq \frac{1}{d-1}.$$

If $|V| < d$, we have $P_H(w) = P_{H-v}(w) = 1$ and the inequality holds. If $|V| = d$, the hypergraph may contain either one edge or no edges. In the former case, we have $P_H(w) = 1 + w(e)$ while $P_{H-v}(w) = 1$ and the inequality reduces to

$$\left| 1 - \frac{1}{1 + w(e)} \right| = \left| \frac{w(e)}{1 + w(e)} \right| \leq \frac{1}{d-1},$$

which obviously holds when $w(e) = 0$. If $w(e) \neq 0$, we can further write

$$\begin{aligned} \left| \frac{w(e)}{1 + w(e)} \right| = \left| \frac{1}{1 + w(e)^{-1}} \right| &\leq \left(\frac{d^d}{(d-1)^{d-1}} - 1 \right)^{-1} = \frac{(d-1)^{d-1}}{d^d - (d-1)^{d-1}} \\ &< \frac{(d-1)^{d-1}}{d^d - d^{d-1}} = \frac{1}{d-1}. \end{aligned}$$

If H contains no edges then $P_{H-v}(w) = P_H(w) = 1$ and the inequality holds as well.

Suppose now $|V| > d$. By the induction hypothesis, for every vertex $v_1 \in V$, we have $P_{H-v_1}(w) \neq 0$. We rewrite (5.5.2.1) as

$$\frac{P_H(w)}{P_{H-v_1}(w)} = 1 + \sum_{\substack{e\in E:\\ v\in e}} w(e)\frac{P_{H-e}(w)}{P_{H-v_1}(w)}. \tag{5.5.2.2}$$

Telescoping, for every edge $e = \{v_1, v_2, \ldots, v_d\}$ containing v_1, we can write

$$\frac{P_{H-e}(w)}{P_{H-v_1}(w)} = \frac{P_{H-e}(w)}{P_{H-\{v_1,v_2,\ldots,v_{d-1}\}}(w)} \cdots \frac{P_{H-\{v_1,v_2\}}(w)}{P_{H-\{v_1\}}(w)}, \tag{5.5.2.3}$$

where by the induction hypothesis, we have

$$P_{H-\{v_1,\ldots,v_{k+1}\}}(w) \neq 0 \quad \text{and}$$
$$\left|1 - \frac{P_{H-\{v_1,\ldots,v_{k+1}\}}(w)}{P_{H-\{v_1,\ldots,v_k\}}(w)}\right| \leq \frac{1}{d-1} \quad \text{for all} \quad k = 1, \ldots, d-1,$$

from which

$$\left|\frac{P_{H-\{v_1,\ldots,v_{k+1}\}}(w)}{P_{H-\{v_1,\ldots,v_k\}}(w)}\right| \leq \frac{d}{d-1} \quad \text{for} \quad k = 1, \ldots, d-1. \tag{5.5.2.4}$$

Combining (5.5.2.2)–(5.5.2.4), we conclude that

$$\left|1 - \frac{P_H(w)}{P_{H-v_1}(w)}\right| \leq \frac{(d-1)^{d-1}}{d^d}\left(\frac{d}{d-1}\right)^{d-1} = \frac{1}{d}, \tag{5.5.2.5}$$

from which it follows that

$$P_H(w) \neq 0.$$

The transformation $z \longmapsto 1/z$ maps the disc

$$D = \left\{z : |1-z| \leq \frac{1}{d}\right\}$$

onto the disc with center on the real axis and whose boundary intersects the real axis in the points $d/(d+1)$ and $d/(d-1)$. Therefore, from (5.5.2.5), we have

$$\left|1 - \frac{P_{H-v_1}(w)}{P_H(w)}\right| \leq \left|1 - \frac{d}{d-1}\right| = \frac{1}{d-1},$$

which completes the induction. □

The bound of Theorem 5.5.2 decreases as $1/ed$ as d grows.

For a weight $w : E \longrightarrow \mathbb{C}$ and a parameter $z \in \mathbb{C}$, let zw denote the weight on the edges of H scaled by z. Then

$$\frac{d^k}{dz^k} P_H(wz)\Big|_{z=0} = k! \sum_{\substack{e_1,\ldots,e_k \\ \text{is a matching}}} w(e_1)\cdots w(e_k)\,.$$

In particular, the derivative can be computed in $|E|^{O(k)}$ time by the direct enumeration of all matchings of k edges in H. As follows from Lemma 2.2.1 and Sect. 2.2.2, for any $0 \le \delta < 1$, fixed in advance, for any complex weights $w : H \longrightarrow \mathbb{C}$ satisfying

$$\sum_{\substack{e\in E: \\ v\in E}} |w(e)| \;\le\; \delta \frac{(d-1)^{d-1}}{d^d}$$

and any $0 < \epsilon < 1$, the value of $P_H(w)$ can be approximated within relative error ϵ in $|E|^{O(\ln|E|-\ln\epsilon)}$ time. If the largest degree of a vertex is bounded above in advance, the computation can be done in genuine polynomial time via the approach of [PR16], see also Sect. 6.6.

The correlation decay approach to computing $P_H(w)$ was tried in [D+14], [S+16]. In particular, a polynomial time approximation algorithm was constructed in [D+14] that counts the number of matchings in a 3-uniform hypergraph such that the degree of every vertex does not exceed 3.

We apply Theorem 5.5.2 to multidimensional permanents, see Sect. 4.4. We show that if each slice of a d-dimensional tensor $A = \left(a_{i_1\ldots i_d}\right)$ is close in the ℓ^1-metric to the tensor of all 1s, then PER $A \neq 0$ and, consequently, ln PER A can be approximated in quasi-polynomial time. This contrasts with Theorem 4.4.2, where we require the deviation to be small in the ℓ^∞-metric.

5.5.3 Theorem. *Let $A = \left(a_{i_1\ldots i_d}\right)$ be an $n \times \ldots \times n$ array of n^d complex numbers, such that*

$$\sum_{1\le i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_d\le n} \left|1 - a_{i_1\ldots i_d}\right| \;\le\; \alpha^{d-1}\frac{(d-1)^{d-1}}{d^d} n^{d-1}$$

for all $1 \le i_j \le n$ and all $j = 1, \ldots, d$, where

$$\alpha \approx 0.2784645428$$

is the positive solution of the equation

$$xe^{1+x} = 1.$$

Then

$$\text{PER } A \neq 0.$$

The following lemma is a weaker version of a bound from [Wa03].

5.5.4 Lemma. *For a positive integer n, let us define a polynomial*

$$p_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}.$$

Then

$$p_n(z) \neq 0 \quad \textit{provided} \quad |z| \leq \alpha n,$$

where $\alpha \approx 0.2784645428$ *is the constant of Theorem 5.5.3.*

Proof. First, we observe that

$$\left|ze^{1-z}\right| \;\leq\; |z|e^{1+|z|} \;\leq\; 1 \quad \text{provided} \quad |z| \leq \alpha.$$

Then for $|z| \leq \alpha$, we have

$$\begin{aligned}\left|1 - e^{-nz} p_n(nz)\right| &= \left|e^{-nz} \sum_{k=n+1}^{\infty} \frac{(nz)^k}{k!}\right| = \left|\left(ze^{(1-z)}\right)^n e^{-n} \sum_{k=n+1}^{\infty} \frac{n^k z^{k-n}}{k!}\right| \\ &\leq e^{-n} \sum_{k=n+1}^{\infty} \frac{n^k}{k!} \;<\; 1\end{aligned}$$

and hence $p_n(nz) \neq 0$. □

Szegő proved that as n grows, the zeros of $p_n(nz)$ converge to the curve

$$\left\{z : \left|ze^{1-z}\right| = 1,\ |z| \leq 1\right\},$$

see [PV97].

5.5.5 Proof of Theorem 5.5.3. We have

$$\text{PER } A = \sum_{\sigma_2,\ldots,\sigma_d \in S_n} \prod_{i=1}^{n} a_{i\sigma_2(i)\ldots\sigma_d(i)} = \sum_{\sigma_2,\ldots,\sigma_d \in S_n} \prod_{i=1}^{n} \left(1 + \left(a_{i\sigma_2(i)\ldots\sigma_d(i)} - 1\right)\right).$$

We consider the complete d-partite graph $H = (V, E)$ with $n+\ldots+n = nd$ vertices and the weight of the edge $(i_1, \ldots, i_d)$ equal $a_{i_1\ldots i_d} - 1$. Let

$$W_k = \sum_{\substack{e_1,\ldots,e_k \in E: \\ e_1,\ldots,e_k \text{ is a matching}}} w(e_1)\cdots w(e_k)$$

be the total weight of k-matchings in H, where we agree that $W_0 = 1$. Then

$$\text{PER } A = \sum_{k=0}^{n} ((n-k)!)^{d-1} W_k = (n!)^{d-1} \sum_{k=0}^{n} \left(\frac{1}{k!}\right)^{d-1} \binom{n}{k}^{-(d-1)} W_k.$$

Let us define the univariate polynomial

$$q(z) = \sum_{k=0}^{n} W_k z^k.$$

Interpreting the value of $q(z)$ as the value of the hypergraph matching polynomial P_H on the scaled weights $z\left(a_{i_1 \ldots i_d} - 1\right)$, from Theorem 5.5.2 we deduce that

$$q(z) \neq 0 \quad \text{provided} \quad |z| \leq \frac{1}{(\alpha n)^{d-1}}.$$

Let

$$p(z) = \sum_{k=0}^{n} \frac{z^k}{k!}.$$

By Lemma 5.5.4,

$$p(z) \neq 0 \quad \text{provided} \quad |z| \leq \alpha n.$$

Applying Corollary 2.5.10 successively to the pairs $\{p, q\}, \{p, p * q\}, \ldots, \{p, p * \ldots * p * q\}$, we conclude that the polynomial

$$r(z) = \sum_{k=0}^{n} \left(\frac{1}{k!}\right)^{d-1} \binom{n}{k}^{-(d-1)} W_k z^k$$

satisfies

$$r(z) \neq 0 \quad \text{provided} \quad |z| \leq 1.$$

In particular,

$$r(1) = \sum_{k=0}^{n} \left(\frac{1}{k!}\right)^{d-1} \binom{n}{k}^{-(d-1)} W_k = (n!)^{-(d-1)} \operatorname{PER} A \neq 0,$$

as claimed. □

Chapter 6
The Independence Polynomial

Known in statistical physics as the partition function of a hard core model, the independence polynomial of a graph is a far-reaching extension of the matching polynomial, demonstrating a much more complicated behavior. The roots of the independence polynomial do not have to be real, but the Dobrushin–Scott–Sokal bound for its complex roots is similar to the bound for the roots of the matching polynomial. The correlation decay is observed for sufficiently small activities but disappears for large activities, so there is phase transition. The highlight of the chapter is in establishing the exact point of that phase transition, first for regular trees, and then, following Weitz, for arbitrary graphs. It also provides us with an instance where the correlation decay approach outperforms the Taylor polynomial interpolation method (so far). The two methods would achieve the same degree of approximation if there are no roots of the independence polynomial near the positive real axis up to the Weitz bound, as was conjectured by Sokal. We prove a result of Regts stating that there are indeed no roots near the positive real axis halfway between the Dobrushin–Scott–Sokal and Weitz bounds.

6.1 The Independence Polynomial of a Graph

6.1.1 Definition. Let $G = (V, E)$ be an undirected graph with set V of vertices, set E of edges, without loops or multiple edges. A set $U \subset V$ of vertices is *independent* if no two vertices of U span an edge of G. We consider the empty set $\emptyset$ independent. Let $\mathbb{C}^V$ be the complex vector space with coordinates indexed by the vertices of G, hence we write $z = (z_v : v \in V)$ for a typical $z \in \mathbb{C}^V$. For a subset $U \subset V$ we consider the monomial

$$\mathbf{z}^U = \prod_{v \in U} z_v,$$

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Algorithms and Combinatorics 30, DOI 10.1007/978-3-319-51829-9_6

where we agree as usual that

$$\mathbf{z}^{\emptyset} = 1.$$

We define the *independence polynomial* of G, $\mathrm{ind} : \mathbb{C}^V \longrightarrow \mathbb{C}$, by

$$\mathrm{ind}_G(z) = \sum_{\substack{U \subset V \\ U \text{is independent}}} \mathbf{z}^U.$$

In particular, $\mathrm{ind}_G(0) = 1$. We call z_v the *activity* of v. In statistical physics, $\mathrm{ind}_G(z)$ is known as the partition function of the "hard core model". It describes mutually repelling particles that can occupy positions at the vertices of a graph and avoid coming too close to each other, that is, avoid occupying adjacent vertices.

Let $v \in V$ be a vertex and let

$$N_v = \{u \in V : \{u, v\} \subset E\}$$

be the *neighborhood* of v in G (note that we do not include v in its own neighborhood). For sets $A, B \subset V$ of vertices, by $G(A)$ we denote the subgraph induced by the subset A of vertices (hence two vertices from A span an edge of $G(A)$ if and only if they span an edge of G) and by $G(A) - B$ we denote the graph obtained from $G(A)$ by deleting all vertices from B together with incident edges. If an independent set U contains v then it cannot contain any of the vertices adjacent to v and we arrive to the identity

$$\mathrm{ind}_G(z) = \mathrm{ind}_{G-v}(z) + z_v \, \mathrm{ind}_{G-v-N_v}(z), \tag{6.1.1.1}$$

see Fig. 6.1 (we use $G - v - N_v$ as a shorthand for $G - (\{v\} \cup N_v)$).

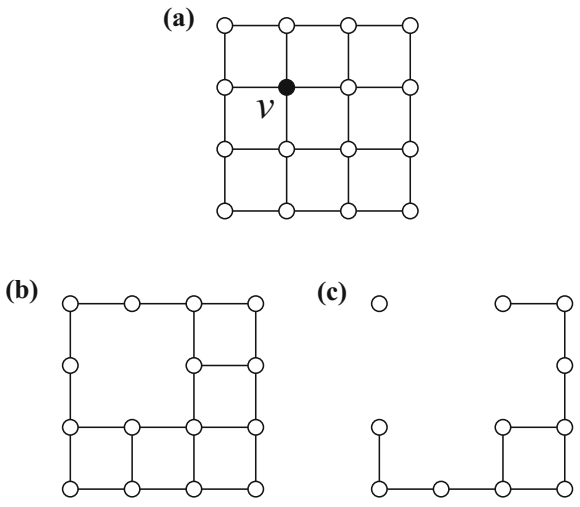


Fig. 6.1 **a** A graph G with a vertex v, **b** the graph $G - v$ and **c** the graph $G - v - N_v$

The following result on the location of zeros of ind_G was obtained by Dobrushin, see [Do96] and [SS05]. We follow [CF16], which, in turn, contains a modification of an argument from [Bo06].

6.1.2 Theorem. *Let $G = (V, E)$ be a graph and let $0 < r_v < 1 : v \in V$ be reals. Suppose that*

$$|z_v| \leq (1 - r_v) \prod_{\substack{u \in V: \\ \{u,v\} \in E}} r_u.$$

Then

$$\mathrm{ind}_G(z) \neq 0.$$

Proof. Recall that for a set $A \subset V$ of vertices, we denote by $G(A)$ the induced subgraph on the set A. We formally consider the polynomial $\mathrm{ind}_{G(A)}$ as a function on $\mathbb{C}^V$, although the variables z_v with $v \notin A$ do not enter into it. We prove by induction on $|A|$ the following two inequalities:

$$\mathrm{ind}_{G(A)}(z) \neq 0 \tag{6.1.2.1}$$

and

$$\left| \frac{\mathrm{ind}_{G(B)}(z)}{\mathrm{ind}_{G(A)}(z)} \right| \leq \left(\prod_{u \in A \setminus B} r_u \right)^{-1} \quad \text{for any} \quad B \subset A \tag{6.1.2.2}$$

and $z \in \mathbb{C}^V$ satisfying the conditions of the theorem. We agree that the right hand side of (6.1.2.2) is 1 if $B = A$. If $A = V$ then (6.1.2.1) is what we need.

If $A = \emptyset$ then (6.1.2.1) and (6.1.2.2) hold trivially. Suppose that $A \neq \emptyset$ and that (6.1.2.1) and (6.1.2.2) hold for all proper subsets of A. Let us choose $v \in A$. By the induction hypothesis, $\mathrm{ind}_{G(A)-v}(z) \neq 0$ and using (6.1.1.1) we can write

$$\begin{aligned} \mathrm{ind}_{G(A)}(z) &= \mathrm{ind}_{G(A)-v}(z) + z_v \, \mathrm{ind}_{G(A)-v-N_v}(z) \\ &= \mathrm{ind}_{G(A)-v}(z) \left(1 + z_v \frac{\mathrm{ind}_{G(A)-v-N_v}(z)}{\mathrm{ind}_{G(A)-v}(z)} \right). \end{aligned} \tag{6.1.2.3}$$

By the induction hypothesis, from (6.1.2.2) it follows that

$$\left| \frac{\mathrm{ind}_{G(A)-v-N_v}(z)}{\mathrm{ind}_{G(A)-v}(z)} \right| \leq \left(\prod_{u \in (A \setminus \{v\}) \setminus (A \setminus (\{v\} \cup N_v))} r_u \right)^{-1} = \left(\prod_{\substack{u \in A: \\ \{u,v\} \in E}} r_u \right)^{-1},$$

where the last equality follows since every vertex $u \in (A \setminus \{v\}) \setminus (A \setminus (\{v\} \cup N_v))$ is necessarily connected to v by an edge, cf. Fig. 6.1. Therefore,

$$\left| z_v \frac{\mathrm{ind}_{G(A)-v-N_v}(z)}{\mathrm{ind}_{G(A)-v}(z)} \right| \leq (1 - r_v) < 1 \tag{6.1.2.4}$$

and (6.1.2.1) follows by (6.1.2.3). We also note that from (6.1.2.3) and the first inequality in (6.1.2.4) it follows that

$$\left| \frac{\mathrm{ind}_{G(A)-v}(z)}{\mathrm{ind}_{G(A)}(z)} \right| \leq r_v^{-1}. \tag{6.1.2.5}$$

Let $B \subset A$ be a subset. If $B = A$ then (6.1.2.2) holds trivially, so we assume that B is a proper subset of A. Then for some $v \in A$ we have $B \subset A \setminus \{v\}$ and applying the induction hypothesis to the pair $B \subset A \setminus \{v\}$ and using (6.1.2.5) we obtain

$$\begin{aligned} \left| \frac{\mathrm{ind}_{G(B)}(z)}{\mathrm{ind}_{G(A)}(z)} \right| &= \left| \frac{\mathrm{ind}_{G(B)}(z)}{\mathrm{ind}_{G(A-v)}(z)} \right| \left| \frac{\mathrm{ind}_{G(A)-v}(z)}{\mathrm{ind}_{G(A)}(z)} \right| \leq \left(\prod_{u \in (A \setminus \{v\}) \setminus \{B\}} r_u \right)^{-1} r_v^{-1} \\ &= \prod_{u \in A \setminus B} r_u^{-1}, \end{aligned}$$

which completes the proof. □

Suppose that the degree of every vertex of G does not exceed some $\Delta \geq 1$. Choosing

$$r_v = \frac{\Delta}{\Delta + 1} \quad \text{for all} \quad v \in V$$

we obtain from Theorem 6.1.2 that

$$\mathrm{ind}_G(z) \neq 0 \quad \text{provided} \quad |z_v| \leq \frac{\Delta^\Delta}{(\Delta+1)^{\Delta+1}} \quad \text{for all} \quad v \in V.$$

Scott and Sokal [SS05] showed that the bound can be improved somewhat.

6.1.3 Theorem. *Suppose that the degree of every vertex of G does not exceed some $\Delta \geq 2$. Then*

$$\mathrm{ind}_G(z) \neq 0 \quad \textit{provided} \quad |z_v| \leq \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta} \quad \textit{for all} \quad v \in V.$$

Proof. The proof is very similar to that of Theorem 5.5.2. We proceed by induction on the number $|V|$ of vertices. If $|V| = 1$, the result holds, so we assume that $|V| > 1$.

We embed into our inductive proof yet another inductive argument (the inner induction as opposed to the outer induction). Namely, we prove by induction on $|V|$ that if $G = (V, E)$ is a graph with the largest degree $\Delta(G) \leq \Delta$ of a vertex and if $v \in V$ is a vertex of degree at most $\Delta - 1$ then

$$\operatorname{ind}_G(z) \neq 0 \quad \text{and} \quad \left|1 - \frac{\operatorname{ind}_{G-v}(z)}{\operatorname{ind}_G(z)}\right| < \frac{1}{\Delta - 1}$$
$$\text{provided} \quad |z_u| \leq \frac{(\Delta - 1)^{\Delta - 1}}{\Delta^{\Delta}} \quad \text{for all} \quad u \in V.$$

The case of $|V| = 1$ is easily verified, so we assume that $|V| \geq 2$. By the outer induction hypothesis, $\operatorname{ind}_{G-v}(z) \neq 0$, so we can rewrite (6.1.1.1) as

$$\frac{\operatorname{ind}_G(z)}{\operatorname{ind}_{G-v}(z)} = 1 + z_v \frac{\operatorname{ind}_{G-v-N_v}(z)}{\operatorname{ind}_{G-v}(z)}. \tag{6.1.3.1}$$

Let $N_v = \{v_1, \ldots, v_k\}$ for some $0 \leq k \leq \Delta - 1$. If $k = 0$, that is, if v is an isolated vertex, then

$$\left|1 - \frac{\operatorname{ind}_{G-v}(z)}{\operatorname{ind}_G(z)}\right| = \left|1 - \frac{1}{1 + z_v}\right| < \frac{1}{\Delta - 1}$$

and the step of the inner induction is completed.

Suppose that $k > 0$ so that v has neighbors in G. Then

$$\frac{\operatorname{ind}_{G-v-N_v}(z)}{\operatorname{ind}_{G-v}(z)} = \frac{\operatorname{ind}_{G-v-v_1}(z)}{\operatorname{ind}_{G-v}(z)} \cdots \frac{\operatorname{ind}_{G-v-v_1-\ldots-v_k}(z)}{\operatorname{ind}_{G-v-v_1-\ldots-v_{k-1}}(z)}. \tag{6.1.3.2}$$

By the inner induction hypothesis

$$\operatorname{ind}_{G-v-v_1-\ldots-v_i}(z) \neq 0 \quad \text{for} \quad i = 1, \ldots, k.$$

Moreover, since the degree of v_i in the graph $G - v - v_1 - \ldots - v_{i-1}$ does not exceed $\Delta - 1$, by the inner induction hypothesis

$$\left|1 - \frac{\operatorname{ind}_{G-v-v_1-\ldots-v_i}(z)}{\operatorname{ind}_{G-v-v_1-\ldots-v_{i-1}}(z)}\right| < \frac{1}{\Delta - 1} \quad \text{for} \quad i = 1, \ldots, k. \tag{6.1.3.3}$$

Hence from (6.1.3.2) we conclude that

$$\left|\frac{\operatorname{ind}_{G-v-N_v}(z)}{\operatorname{ind}_{G-v}(z)}\right| < \left(\frac{\Delta}{\Delta - 1}\right)^{\Delta - 1}$$

and from (6.1.3.1) we conclude that

$$\left|1 - \frac{\operatorname{ind}_G(z)}{\operatorname{ind}_{G-v}(z)}\right| < \frac{(\Delta - 1)^{\Delta - 1}}{\Delta^{\Delta}} \cdot \frac{\Delta^{\Delta - 1}}{(\Delta - 1)^{\Delta - 1}} = \frac{1}{\Delta}.$$

Therefore, $\mathrm{ind}_G(z) \neq 0$ and, as in the proof of Theorem 5.5.2, we conclude that

$$\left|1 - \frac{\mathrm{ind}_{G-v}(z)}{\mathrm{ind}_G(z)}\right| < \frac{1}{\Delta - 1},$$

which concludes the inner induction.

To conclude the outer induction, it remains to prove that $\mathrm{ind}_G(z) \neq 0$ if the degree of every vertex v of G is Δ. We choose an arbitrary vertex v and use (6.1.3.1) and (6.1.3.2) as above, only that the right hand side of (6.1.3.2) is a product of Δ (as opposed to $\Delta - 1$) factors. Still, the degree of v_i in $G - v - v_1 - \ldots - v_{i-1}$ does not exceed $\Delta - 1$ and therefore (6.1.3.3) still holds. Hence from (6.1.3.2) we conclude that

$$\left|\frac{\mathrm{ind}_{G-v-N_v}(z)}{\mathrm{ind}_{G-v}(z)}\right| < \left(\frac{\Delta}{\Delta - 1}\right)^{\Delta}$$

and from (6.1.3.1) we have

$$\left|1 - \frac{\mathrm{ind}_G(z)}{\mathrm{ind}_{G-v}(z)}\right| < \frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta}} \cdot \frac{\Delta^{\Delta}}{(\Delta-1)^{\Delta}} = \frac{1}{\Delta - 1}$$

and $\mathrm{ind}_G(z) \neq 0$. □

As is discussed in [SS05], the bound of Theorem 6.1.3 is optimal, as it is asymptotically achieved on regular trees. Also, see [SS05] for extensions, generalizations and connections to Lovász's Local Lemma.

6.1.4 Example: the Tutte polynomial of a graph. In [CF16], Csikvári and and Frenkel deduced from Theorem 6.1.2 bounds on the zeros of a wide class of graph polynomials, which they call polynomials of exponential type. We consider one example from [CF16], the Tutte polynomial of a graph.

Let $G = (V, E)$ be a graph. Let $w = (w_e : e \in E)$ be a vector of complex variables indexed by the edges e of G and let ζ be yet another complex variable. We define the *Tutte polynomial* of G by

$$T_G(\zeta, w) = \sum_{A \subset E} \zeta^{\kappa(A)} \prod_{e \in A} w_e,$$

where the sum is taken over all sets A of edges of G and $\kappa(A)$ is the number of connected components in the graph with set V of vertices and set A of edges. In particular, T_G is a monic polynomial in ζ of degree $|V|$ since for $A = \emptyset$ the corresponding monomial is just $\zeta^{|V|}$.

We express $T_G(\zeta, w)$ in terms of the independence polynomial of some other graph $\widehat{G} = (\widehat{V}, \widehat{E})$ as follows. The vertex set $\widehat{V}$ consists of all subsets $U \subset V$ such that $|U| \geq 2$. Two subsets U_1 and U_2 span an edge in $\widehat{G}$ if and only if $U_1 \cap U_2 \neq \emptyset$. Hence the independent sets in $\widehat{G}$ are the collections of pairwise disjoint subsets

$U_1, \ldots, U_k \subset V$, each of cardinality at least 2. Let $G(U)$ be the subgraph of G induced on U. We define the activity z_U of a vertex U of $\widehat{G}$ by

$$z_U = 0 \quad \text{if} \quad G(U) \quad \text{is not connected}$$

and by

$$z_U = \zeta^{1-|U|} \prod_{\substack{e \in E \\ \text{both endpoints of } e \text{ lie in } U}} w_e \quad \text{if} \quad G(U) \quad \text{is connected.}$$

If $U_1, \ldots, U_k \subset V$ are pairwise disjoint subsets of cardinality at least 2, such that all induced subgraphs $G(U_1), \ldots, G(U_k)$ are connected then for the set $A \subset E$ of edges that is the union of the sets of edges in $G(U_1), \ldots, G(U_k)$, we have

$$\kappa(A) = k + \left(|V| - \sum_{i=1}^{k} |U_i| \right),$$

since the connected components of the graph with set V of vertices and set A of edges are the induced subgraphs $G(U_1), \ldots, G(U_k)$ and the remaining isolated vertices. On the other hand,

$$\sum_{i=1}^{k} (1 - |U_i|) = k - \sum_{i=1}^{k} |U_i|,$$

from which we deduce that

$$T_G(\zeta, w) = \zeta^{|V|} \operatorname{ind}_{\widehat{G}}(z). \tag{6.1.4.1}$$

Let us consider constant weights $w_e = w_0$ for some $w_0 \in \mathbb{C}$. Using (6.1.4.1) and Theorem 6.1.2, Csikvári and and Frenkel [CF16] prove that $T_G(\zeta, w) \neq 0$ if

$$|\zeta| > \gamma \Delta(G) (1 + |w_0|)^{\Delta(G)}$$

for some absolute constant $\gamma > 0$ (one can choose $\gamma = 21$).

For some specializations of the Tutte polynomial better bounds are known. For example, if $w_e = -1$ for all $e \in E$ then $\operatorname{chr}_G(\zeta) = T_G(\zeta, w)$ is the *chromatic polynomial* of G, which, for positive integer ζ counts the number of ways to color the vertices of G into at most ζ colors so that no two vertices spanning an edge of G are colored with the same color, see also Lemma 6.5.5. Sokal [S01a] proved that $\operatorname{chr}_G(\zeta) \neq 0$ if $|\zeta| > 8\Delta(G)$.

6.1.5 Computing the independence polynomial. As was noticed in [Re15], Theorem 6.1.3 allows one to approximate $\operatorname{ind}_G(z)$ within relative error ϵ in $|V|^{O(\ln |V| - \ln \epsilon)}$ time provided

$$|z_v| \leq \delta \frac{(\Delta - 1)^{\Delta - 1}}{\Delta^{\Delta}}$$

for any $0 < \delta < 1$, fixed in advance, where $\Delta = \Delta(G) \geq 2$ is the largest degree of vertex of G. To see that, let us consider a univariate function $f(\zeta) = \ln \mathrm{ind}_G(\zeta z)$. Let $p_m(\zeta)$ be the Taylor polynomial of degree m of $f(\zeta)$ at $\zeta = 0$. It follows from Lemma 2.2.1 that $p_m(1)$ approximates $f(1)$ within an additive error ϵ provided $m = O(\ln |V| - \ln \epsilon)$. Moreover, by Sect. 2.2.2, to compute the $p_m(\zeta)$ it suffices to compute

$$\frac{d^k}{d\zeta^k} \mathrm{ind}_G(\zeta z)\Big|_{\zeta=0},$$

which in turn reduces to the enumeration of all independent sets of G of size at most m, which can be accomplished in $|V|^{O(m)}$ time. Patel and Regts show [PR16] that if the largest degree $\Delta(G)$ of a vertex of G is fixed in advance then $\mathrm{ind}_G(z)$ can be approximated in polynomial time $(|V|/\epsilon)^{O(1)}$, see Sect. 6.6.

Similar algorithms are described in [PR16] and [Re15] for other combinatorial polynomials. As Regts notes [Re15], for some polynomials p computing values $p(z)$ for with $|z|$ *large* is feasible, for which one should apply Lemma 2.2.1 to the polynomial

$$\tilde{p}(z) = z^{\deg p} p(1/z).$$

A natural example is provided by the chromatic polynomial, see Sect. 6.1.4 and Lemma 6.5.5, where Lemma 2.2.1 produces a quasi-polynomial approximation algorithm to compute $\mathrm{chr}_G(\zeta)$ provided $|\zeta| > \gamma\Delta(G)$ for any $\gamma > 8$, fixed in advance. Sokal conjectured, see [Ja03], that $\mathrm{chr}_G(\zeta) \neq 0$ provided $\Re\,\zeta > \Delta(G)$. Should this conjecture be true, $\mathrm{chr}_G(\zeta)$ can be efficiently approximated provided $\Re\,\zeta > (1+\delta)\Delta(G)$ for any fixed $\delta > 0$, see [PR16].

We note that

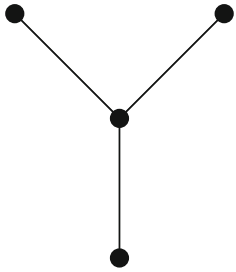
$$\frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta}} = \frac{1}{\Delta e}\left(1 + O\left(\frac{1}{\Delta}\right)\right) \quad \text{as} \quad \Delta \longrightarrow +\infty. \tag{6.1.5.1}$$

It is shown in [LV99] that the problem of approximating $\mathrm{ind}_G(z)$ is NP-hard provided $z = (\lambda, \ldots, \lambda)$ for $\lambda > c/\Delta(G)$, where $c > 0$ is an absolute constant.

There are certain parallels between the matching polynomial considered in Chap. 5 and the independence polynomial. Given a graph $G = (V, E)$, one can consider its *line graph* $L(G)$. The vertices of $L(G)$ are the edges of E and two vertices of $L(G)$ span an edge if the corresponding edges in G share a vertex. Then a matching in G corresponds to an independent set in $L(G)$ and vice versa. Line graphs form a rather restricted class of graphs, for example, they are always *claw-free*, that is, do not contain an induced subgraph pictured on Fig. 6.2.

Extending Theorem 5.1.2, Chudnovsky and Seymour [CS07] proved that the roots of the univariate independence polynomial (when all activities z_v are equal) of a claw-free graph are real. In that case, using Lemma 2.2.3 and arguing as in Sect. 5.1.7, for *any* $\delta \geq 1$, fixed in advance and any complex z such that

Fig. 6.2 A claw



$$|z| \le \delta \frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta}} \quad \text{and} \quad |\pi - \arg z| \ge \frac{1}{\delta}$$

we can approximate ind_G at $z_v = z$ for all $v \in V$ within a relative error $\epsilon > 0$ in $n^{O(\ln n - \ln \epsilon)}$ time. Furthermore, Patel and Regts show [PR16] that if $\Delta(G)$ is fixed in advance, the algorithm can be made genuinely polynomial, see also Sect. 6.6.

We also note that the nearest to the origin complex root of the univariate independence polynomial of any graph is necessarily negative real [SS05]. More precisely, let us fix any vector of non-negative real activities $x = (x_v \ge 0 : v \in V)$ at the vertices V of a graph G and for a $\zeta \in \mathbb{C}$, let us consider its scaling

$$\zeta x = (\zeta x_v : v \in V)\,.$$

Then among the roots of the univariate polynomial

$$g(z) = \mathrm{ind}_G(\zeta x) \quad \text{where} \quad \zeta x = (\zeta x_v : v \in V)$$

nearest to the origin, there is necessarily a negative real root. We prove this later in Theorem 6.5.4.

6.2 The Independence Polynomial of Regular Graphs

6.2.1 The probability space of independent sets. Let $G = (V, E)$ be a graph. For a real $t > 0$ we consider the value of the independence polynomial $\mathrm{ind}_G(z)$ where $z_v = t$ for all $v \in V$, which we denote just by $\mathrm{ind}_G(t)$. We consider the set of all independent sets $S \subset V$, including the empty set, as a finite probability space with

$$\Pr(S) = \frac{t^{|S|}}{\mathrm{ind}_G(t)}.$$

Then

$$t\frac{d}{dt}\ln \mathrm{ind}_G(t) = \frac{t\,\mathrm{ind}'_G(t)}{\mathrm{ind}_G(t)} = \frac{1}{\mathrm{ind}_G(t)} \sum_{\substack{S\subset V \\ \text{is independent}}} |S| t^{|S|}$$

$$= \sum_{\substack{S\subset V \\ \text{is independent}}} |S| \Pr(S) = \mathbf{E}|S| \qquad (6.2.1.1)$$

is naturally interpreted as the expected size of a random independent set S. Consequently,

$$\frac{t}{|V|}\frac{d}{dt}\ln \mathrm{ind}_G(t)$$

is naturally interpreted as the expected fraction of vertices contained in a random independent set S.

Assume now that G is k-regular, that is, every vertex of G is incident to exactly k edges of G. Davies, Jenssen, Perkins and Roberts proved [D+15] that the expected fraction of vertices contained in a random independent set of a k-regular graph is maximized when G is the vertex-disjoint union of k-regular complete bipartite graphs, cf. Fig. 5.11. We follow their proof below, see also [Zh16] for a survey.

6.2.2 Theorem. *Let $G = (V, E)$ be a k-regular graph. Then for any $t > 0$ we have*

$$\frac{t}{|V|}\frac{d}{dt}\ln \mathrm{ind}_G(t) \;\le\; \frac{t(1+t)^{k-1}}{2(1+t)^k - 1},$$

where equality is attained if and only if G is the vertex-disjoint union of k-regular complete bipartite graphs. Consequently,

$$\mathrm{ind}_G(t) \;\le\; \left(2(1+t)^k - 1\right)^{\frac{|V|}{2k}},$$

where equality is attained if and only if G is the vertex-disjoint union of k-regular complete bipartite graphs.

Following [D+15], we start with a lemma.

6.2.3 Lemma. *Let $G = (V, E)$ be a graph where $0 < |V| \le n$. Then for $t > 0$ we have*

$$\frac{\mathrm{ind}'_G(t)}{\mathrm{ind}_G(t) - 1} \;\le\; \frac{n(1+t)^{n-1}}{(1+t)^n - 1},$$

with equality obtained if and only if $E = \emptyset$ so that G consists of n isolated vertices.

Proof. First, let us assume that $|V| = n$. Let $G^\circ = (V, \emptyset)$ be the graph with set V of vertices and no edges. Then every set $S \subset V$ of vertices is independent and

$$\mathrm{ind}_{G^\circ}(t) = \sum_{m=0}^{n} \binom{n}{m} t^m = (1+t)^n.$$

Let

$$\operatorname{ind}_G(t) = \sum_{m=0}^{n} a_m(G)t^m,$$

where a_m is the number of independent m-sets in G. Since each independent $(m+1)$-set in G contains exactly $m+1$ independent m-sets and any independent m-set in G is contained in at most $|V| - m$ independent sets of size $m+1$, we obtain

$$(m+1)a_{m+1} \;\le\; (n-m)a_m \quad \text{for} \quad m = 0, 1, \ldots, n-1 \tag{6.2.3.1}$$

and, consequently,

$$\frac{a_{m+1}}{\binom{n}{m+1}} \;\le\; \frac{a_m}{\binom{n}{m}} \quad \text{for} \quad m = 0, \ldots, n-1.$$

Iterating, we obtain $a_j/\binom{n}{j} \le a_i/\binom{n}{i}$ provided $j \ge i$, which we write as

$$a_j\binom{n}{i} \;\le\; a_i\binom{n}{j} \quad \text{provided} \quad n \ge j \ge i \ge 0. \tag{6.2.3.2}$$

Let

$$t\,\operatorname{ind}'_{G^\circ}(t)\,(\operatorname{ind}_G(t) - 1) = \sum_{m=2}^{2n} b_m(G)t^m \quad \text{and}$$
$$t\,\operatorname{ind}'_{G}(t)\,(\operatorname{ind}_{G^\circ}(t) - 1) = \sum_{m=2}^{2n} c_m(G)t^m,$$

where

$$b_m = \sum_{\substack{i+j=m\\ i,j>0}} j\binom{n}{j}a_i \quad \text{and} \quad c_m = \sum_{\substack{i+j=m\\ i,j>0}} i\,a_i\binom{n}{j}.$$

Hence

$$b_m - c_m = \sum_{\substack{i+j=m\\ i,j>0}} a_i\binom{n}{j}(j-i) = \sum_{\substack{i+j=m\\ j>i>0}} (j-i)\left(a_i\binom{n}{j} - a_j\binom{n}{i}\right) \ge 0$$

by (6.2.3.2). In addition, $b_m = c_m$ for all m if and only if (6.2.3.1) holds for all m. Hence

$$t\,\operatorname{ind}'_{G^\circ}(t)\,(\operatorname{ind}_G(t) - 1) \;\ge\; t\,\operatorname{ind}'_{G}(t)\,(\operatorname{ind}_{G^\circ}(t) - 1) \quad \text{for all} \quad t > 0$$

with equality if and only if $G = G^\circ$ and the proof follows assuming that $|V| = n$.

Since for $n \geq 2$, we have

$$\frac{(1+t)^n - 1}{n(1+t)^{n-1}} - \frac{(1+t)^{n-1} - 1}{(n-1)(1+t)^{n-2}} = \frac{1 + nt - (1+t)^n}{n(n-1)(1+t)^{n-1}} < 0,$$

we conclude that

$$\frac{n(1+t)^{n-1}}{(1+t)^n - 1} > \frac{(n-1)(1+t)^{n-2}}{(1+t)^{n-1} - 1}$$

and the proof follows for any $|V| \leq n$. □

Given an independent set $S \subset V$ in G, we call a vertex $v \in V$ *occupied* by S if $v \in S$ and *unoccupied* otherwise. A vertex $v \in V$ is called *uncovered* by S if it is not adjacent to any occupied vertex and *covered* otherwise. In particular, an occupied vertex is necessarily uncovered but an uncovered vertex may or may not be occupied. The set of neighbors of v that are not adjacent to any vertex $u \in S$ that is not a neighbor of v is called the *free neighborhood* of v (the vertex v is not a neighbor of itself). Vertices in the free neighborhood may or may not be covered, see Fig. 6.3.

As in Sect. 6.2.1, we consider the set of independent sets in G as a probability space.

6.2.4 Lemma. *Let $v \in V$ be a vertex, let p_v be the probability that v is occupied and let q_v be the probability that v is uncovered. Then*

1. *We have*
$$p_v = \frac{t}{1+t} q_v.$$

2. *Let us fix a set U of neighbors of v such that the probability that U is a free neighborhood of v with respect to an independent set is positive and let H be the subgraph induced by U. Then the conditional probability that v is uncovered given that U is the free neighborhood of v is $1/\operatorname{ind}_H(t)$, where we agree that the ratio is 1 if U is empty.*

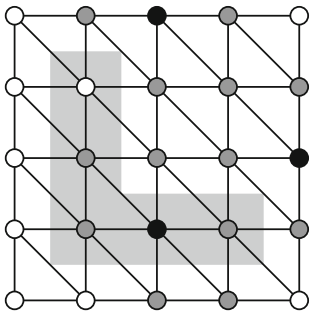


Fig. 6.3 An independent set (*black dots*), covered vertices (*grey dots*), uncovered vertices (*white dots*) and the free neighborhood of the central vertex (dots inside the shaded region)

3. *Let U and H be as in Part (2). Then the conditional expectation of $|U \cap S|$ given that U is the free neighborhood of v is $t\ \mathrm{ind}'_H(t)/\mathrm{ind}_H(t)$, where we agree that the ratio is 0 if U is empty.*

Proof. In Part (1), if v is unoccupied and uncovered by an independent set S then $S' = S \cup \{v\}$ is an independent set, $\mathbf{Pr}(S') = t\mathbf{Pr}(S)$ and v is occupied by S'. Similarly, if v is occupied by S then $S' = S \setminus \{v\}$ is an independent set, $\mathbf{Pr}(S') = t^{-1}\mathbf{Pr}$ and v is uncovered by S. Consequently,

$$q_v = \sum_{S: v \text{ is uncovered}} \mathbf{Pr}(S) = \sum_{S: v \text{ is occupied}} \mathbf{Pr}(S) + \sum_{\substack{S: v \text{ is uncovered} \\ \text{and unoccupied}}} \mathbf{Pr}(S)$$
$$= \sum_{S: v \text{ is occupied}} \mathbf{Pr}(S) + t^{-1} \sum_{S: v \text{ is occupied}} \mathbf{Pr}(S) = \frac{1+t}{t} p_v$$

and the proof of Part (1) follows.

In Part (2), if $U = \emptyset$ then then every neighbor u of v is covered by a vertex that is not a neighbor of v and hence $u \notin S$ and v is necessarily uncovered. Suppose now that $U \neq \emptyset$ and let Σ be the set of independent sets S for which U is the free neighborhood of v. Then, for $S \in \Sigma$ the vertex v is uncovered if and only if $S \cap U = \emptyset$. If $S \in \Sigma$ is an independent set then $S_1 = S \cap U$ is an independent set in H, $S_2 = S \setminus U$ is an independent set in G such that $S_2 \in \Sigma$ and $\mathbf{Pr}(S) = t^{|S_1|}\mathbf{Pr}(S_2)$. Vice versa, if $S_1 \subset U$ is an independent set in H and $S_2 \in \Sigma$ is an independent set such that $S_2 \cap U = \emptyset$ then $S = S_1 \cup S_2$ is an independent set such that $S \in \Sigma$ and $\mathbf{Pr}(S) = \mathbf{Pr}(S_2)t^{|S_1|}$. Hence

$$\sum_{S \in \Sigma} \mathbf{Pr}(S) = \sum_{\substack{S_1:\ S_1 \text{ is independent in } H \\ S_2 \in \Sigma:\ S_2 \cap U = \emptyset}} t^{S_1}\mathbf{Pr}(S_2)$$
$$= \mathrm{ind}_H(t) \sum_{S_2 \in \Sigma:\ S_2 \cap U = \emptyset} \mathbf{Pr}(S_2) \tag{6.2.4.1}$$

and

$$\mathbf{Pr}\big(S \in \Sigma :\ S \cap U = \emptyset \big| S \in \Sigma\big) = \left(\sum_{S \in \Sigma:\ S \cap U = \emptyset} \mathbf{Pr}(S)\right) \Big/ \left(\sum_{S \in \Sigma} \mathbf{Pr}(S)\right)$$
$$= \frac{1}{\mathrm{ind}_H(t)}$$

and the proof of Part (2) follows.

To prove Part (3), we define Σ as above. Clearly, if $U = \emptyset$ then the conditional expectation of $|U \cap S|$ is 0. We assume therefore that $U \neq \emptyset$. Arguing as in the proof of Part (2), we obtain

$$\sum_{S\in\Sigma}|U\cap S|\cdot\mathbf{Pr}(S)=\sum_{\substack{S_1:\ S_1 \text{ is independent in } H\\ S_2\in\Sigma:\ S_2\cap U=\emptyset}}|S_1|t^{|S_1|}\mathbf{Pr}(S_2)$$
$$=t\ \mathrm{ind}'_H(t)\sum_{S_2\in\Sigma:\ S_2\cap U=\emptyset}\mathbf{Pr}(S_2)$$

and by (6.2.4.1)

$$\mathbf{E}\left(|S\cap U|:\ S\in\Sigma\right)=\left(\sum_{S\in\Sigma}|S\cap U|\cdot\mathbf{Pr}(S)\right)\Big/\left(\sum_{S\in\Sigma}\mathbf{Pr}(S)\right)$$
$$=\frac{t\ \mathrm{ind}'_H(t)}{\mathrm{ind}_H(t)},$$

which concludes the proof of Part (3). □

Now we are ready to prove Theorem 6.2.2.

6.2.5 Proof of Theorem 6.2.2. As before, we consider the set of all independent sets in G as a probability space. For a vertex $v\in V$, let p_v be the probability that the vertex is occupied and let q_v be the probability that the vertex is uncovered. Let N_v be the neighborhood of v in G and let $U_{v,S}$ be the free neighborhood of v with respect to an independent set S.

Let $\mathcal{U}_v$ be the set of all subsets $U\subset V$ that appear as the free neighborhood of v with positive probability $y_{v,U}$, so that

$$\sum_{U\in\mathcal{U}_v}y_{v,U}=1\quad\text{for all}\quad v\in V.$$

Let

$$\mathcal{U}=\bigcup_{v\in V}\mathcal{U}_v$$

and for $U\in\mathcal{U}$ let

$$x_U=\sum_{v:\ U\in\mathcal{U}_v}y_{v,U}.$$

Hence

$$\sum_{U\in\mathcal{U}}x_U=\sum_{v\in V}\sum_{U\in\mathcal{U}_v}y_{v,U}=|V|. \tag{6.2.5.1}$$

Let $G(U)$ denote the subgraph induced by U.

Using Part (1) of Lemma 6.2.4, we express the average size of a random independent set S as follows:

$$\mathbf{E}|S|=\sum_{v\in V}p_v=\frac{t}{1+t}\sum_{v\in V}q_v.$$

From Part (2) of Lemma 6.2.4 we further write

$$\sum_{v\in V} q_v = \sum_{v\in V}\sum_{U\in\mathcal{U}_v} \frac{y_{v,U}}{\mathrm{ind}_{G(U)}(t)} = \sum_{U\in\mathcal{U}} \frac{x_U}{\mathrm{ind}_{G(U)}(t)}.$$

Hence

$$\mathbf{E}|S| = \frac{t}{1+t}\sum_{U\in\mathcal{U}} \frac{x_U}{\mathrm{ind}_{G(U)}(t)}. \tag{6.2.5.2}$$

On the other hand, since every vertex $u \in S$ has k neighbors, for any independent set S we can write

$$|S| = \frac{1}{k}\sum_{v\in V} |N_v \cap S| = \frac{1}{k}\sum_{v\in V} |U_{v,S} \cap S|.$$

Using Part (3) of Lemma 6.2.4, we write

$$\mathbf{E}|S| = \frac{1}{k}\sum_{\substack{v\in V\\ U\in\mathcal{U}_v}} \frac{t\,\mathrm{ind}'_{G(U)}(t)}{\mathrm{ind}_{G(U)}(t)} y_{v,U} = \frac{1}{k}\sum_{U\in\mathcal{U}} \frac{t\,\mathrm{ind}'_{G(U)}(t)}{\mathrm{ind}_{G(U)}(t)} x_U. \tag{6.2.5.3}$$

Since $|U| \le k$, from Lemma 6.2.3 we have

$$\frac{t}{k}\,\mathrm{ind}'_{G(U)}(t) \;\le\; \frac{t(1+t)^{k-1}}{(1+t)^k-1}\left(\mathrm{ind}_{G(U)}(t) - 1\right)$$

and hence using that $x(U) > 0$ we obtain from (6.2.5.3)

$$\mathbf{E}|S| \;\le\; \frac{t(1+t)^{k-1}}{(1+t)^k-1}\sum_{U\in\mathcal{U}} x(U) - \frac{t(1+t)^{k-1}}{(1+t)^k-1}\sum_{U\in\mathcal{U}} \frac{x(U)}{\mathrm{ind}_{G(U)}(t)}.$$

Using (6.2.5.2), we conclude that

$$\mathbf{E}|S| \;\le\; \frac{t(1+t)^{k-1}}{(1+t)^k-1}\sum_{U\in\mathcal{U}} x(U) - \frac{(1+t)^k}{(1+t)^k-1}\mathbf{E}|S|,$$

so that

$$\mathbf{E}|S| \;\le\; \frac{t(1+t)^{k-1}}{2(1+t)^k-1}\sum_{U\in\mathcal{U}} x(U).$$

Applying (6.2.5.1), we obtain

$$\frac{1}{|V|}\mathbf{E}|S| \;\le\; \frac{t(1+t)^{k-1}}{2(1+t)^{k-1}}. \tag{6.2.5.4}$$

The desired inequality follows by (6.2.1.1). We get equality in (6.2.5.4) if every free neighborhood that appears with positive probability consists of exactly k disconnected points or empty. □

6.3 Correlation Decay for Regular Trees

6.3.1 Occupancy probabilities on graphs and trees. Let $G = (V, E)$ be a graph and let $z = \big(z_v : v \in V\big)$ be a vector of non-negative activities. We consider the set of independent sets S in a given graph G as a finite probability space where

$$\mathbf{Pr}(S) = (\mathrm{ind}_G(z))^{-1} \prod_{v \in S} z_v$$

(if $S = \emptyset$ then the corresponding product is 1). Let $p(v)$ be the probability that a vertex v is occupied, that is, belongs to a random independent set S. We rewrite (6.1.1.1) as

$$\frac{\mathrm{ind}_{G-v}(z)}{\mathrm{ind}_G(z)} = \frac{1}{1 + z_v \frac{\mathrm{ind}_{G-v-N_v}(z)}{\mathrm{ind}_{G-v}(z)}}. \tag{6.3.1.1}$$

Then

$$1 - p(v) = \frac{\mathrm{ind}_{G-v}(z)}{\mathrm{ind}_G(z)}$$

is the probability that a random independent set S in G does not contain v.

If G is a tree then $G - v$ is a vertex-disjoint union of trees and hence the ratio

$$\frac{\mathrm{ind}_{G-v-N_v}(z)}{\mathrm{ind}_{G-v}(z)}$$

is naturally interpreted as the probability that none of the neighbors of v is occupied in each of the trees obtained from G by deleting v.

First, we consider the case of an (almost) regular tree $\mathbb{T}_n^k$, see Sect. 5.2.4, in detail.

6.3.2 Trees $\mathbb{T}_n^k$ and the phase transition. Let us consider a tree $\mathbb{T}_n^k$, with vertices at the levels $0, 1, \ldots, n$, with one vertex, called the root, at the 0th level connected to $(k-1)$ vertices at the level 1 and with every vertex at the i-th level connected to one vertex at the $(i-1)$-st level and $k-1$ vertices at the $(i+1)$-st level, for $i = 1, \ldots, n-1$, see Sect. 5.2.4 (we assume that $k \geq 3$). If a vertex v at the i-th level is connected to a vertex u at the $(i+1)$-st level, we call u a *descendant* of v.

We fix a $t > 0$ and, as in Sect. 6.3.1 consider the set of all independent sets in $\mathbb{T}_n^k$ as a probability space, with probability of an independent set S proportional to $t^{|S|}$. In other words, we set all activities $z_v = t$. Let $p_n = p_{k,n}(t)$ be the probability that

root is occupied, that is, lies in a random independent set of $\mathbb{T}_n^k$. We are interested in the asymptotic behavior of p_n when k and t are fixed and n grows.

The equation (6.3.1.1) implies that

$$1 - p_n = \frac{1}{1 + t\,(1 - p_{n-1})^{k-1}} \quad \text{where} \quad p_0 = \frac{t}{1+t}. \tag{6.3.2.1}$$

It turns out that the asymptotic behaviors of p_n for large and small t are very different. Namely, let

$$t_c = \frac{(k-1)^{k-1}}{(k-2)^k},$$

called the *critical* t. Then for $t < t_c$ there exists

$$p_\infty = \lim_{n \longrightarrow \infty} p_n$$

while for $t > t_c$ there exist limits

$$p_{\text{even}} = \lim_{n \longrightarrow \infty} p_{2n} \quad \text{and} \quad p_{\text{odd}} = \lim_{n \longrightarrow \infty} p_{2n+1}$$

and $p_{\text{even}} \neq p_{\text{odd}}$. The values $t < t_c$ are called *subcritical* whereas values $t > t_c$ are called *supercritical*. Physicists say that the model experiences a *phase transition* at $t = t_c$.

In view of (6.3.2.1), the results follow from the following general theorem, cf. [Sp75].

6.3.3 Theorem. *Fix some $t > 0$ and an integer $k > 2$ and consider the transformation*

$$T(x) = T_{t,k}(x) = \frac{1}{1 + tx^{k-1}} \quad \textit{for} \quad 0 \le x \le 1.$$

Let

$$t_c = \frac{(k-1)^{k-1}}{(k-2)^k}.$$

For a positive integer n, let T^n denote the n-th iteration of T, so that $T^2(x) = T(T(x))$, $T^3(x) = T(T(T(x))$, etc.

Then there exists a unique point $x_0 = x_0(t,k)$ such that $T(x_0) = x_0$. If $t < t_c$ then

$$\lim_{n \longrightarrow \infty} T^n(x) = x_0 \quad \textit{for all} \quad 0 \le x \le 1.$$

Moreover, the convergence is exponentially fast, meaning that there exist $\gamma = \gamma(t,k) > 0$ and $0 < \delta = \delta(t,k) < 1$ such that

$$\left|\ln T^n(x) - \ln x_0\right| \ \le \ \gamma\delta^n \quad \textit{for all} \quad 0 \le x \le 1.$$

If $t > t_c$ then there exist $x_- = x_-(t,k)$ and $x_+ = x_+(t,k)$ such that

$$x_- < x_0 < x_+$$

while

$$\lim_{n\longrightarrow\infty} T^{2n}(x) = x_- \quad \textit{for all} \quad 0 \le x < x_0$$

and

$$\lim_{n\longrightarrow\infty} T^{2n}(x) = x_+ \quad \textit{for all} \quad x_0 < x \le 1.$$

Proof. It is convenient to parameterize $x = e^{-s}$ for $0 \le s \le +\infty$. In the new coordinates, T can be written as

$$T(s) = \ln\left(1 + te^{-s(k-1)}\right).$$

Since $T(s)$ is decreasing from $T(0) = \ln(1+t) > 0$ to $T(+\infty) = 0$, there is a unique fixed point $a = a(t)$ such that $T(a) = a$, see Fig. 6.4.

Moreover, if $s > a$ then $T(s) < T(a) = a$ and if $s < a$ then $T(s) > T(a) = a$. Since for $s > 0$ we have $T_{t_1}(x) > T_{t_2}(x)$ if and only if $t_1 > t_2$ we conclude that $a(t)$ is an increasing continuous function of t. In addition,

$$\lim_{t\longrightarrow 0+} a(t) = 0 \quad \text{and} \quad \lim_{t\longrightarrow +\infty} a(t) = +\infty$$

and hence the set of possible values of $a(t)$ is the interval $(0, +\infty)$.

We have

$$T'(s) = -\frac{t(k-1)e^{-s(k-1)}}{1 + te^{-s(k-1)}}$$

and

$$T'(a) = -\frac{t(k-1)e^{-a(k-1)}}{1 + te^{-a(k-1)}}.$$

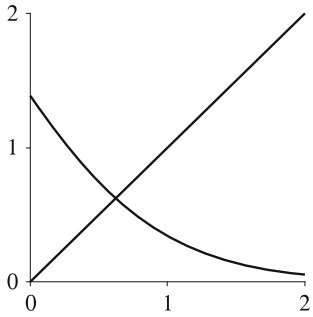


Fig. 6.4 The graphs of $y = \ln\left(1 + 3e^{-2x}\right)$ and $y = x$.

Since

$$1 + te^{-a(k-1)} = e^a,$$

we conclude that

$$t = e^{a(k-1)}\left(e^a - 1\right) \quad \text{and} \quad T'(a) = -(k-1)\left(1 - e^{-a}\right).$$

If $a = \ln \frac{k-1}{k-2}$ then $T'(a) = -1$ and

$$t = \left(\frac{k-1}{k-2}\right)^{k-1} \frac{1}{k-2} = \frac{(k-1)^{k-1}}{(k-2)^k} = t_c.$$

Since $a(t)$ is an increasing function of t, we conclude that for $t < t_c$ we have $0 > T'(a) > -1$ and for $t > t_c$ we have $T'(a) < -1$.

It is now clear that if $t > t_c$ then a is an unstable fixed point: if $s \neq a$ is sufficiently close to a then $|T(s) - a| > |s - a|$ and hence for any $s \neq 0$ the sequence $T^n(s)$ cannot converge to a. On the other hand, if $t < t_c$ then a is a locally stable fixed point: if s is sufficiently close to a then $|T(s) - a| \leq \delta |x - a|$ for some $0 < \delta < 1$ and for any s sufficiently close to a the sequence $T^n(s)$ converges to a.

We consider the second iteration $T(T(s))$. Clearly, $T(T(s))$ is an increasing function of s. We claim that $T(T(s))$ is either concave or has exactly one inflection point, where it changes from convex to concave, see Fig. 6.5.

We have

$$\begin{aligned}(T(T(s))' =& T'(T(s))T'(s)\\ =& \left(-\frac{t(k-1)\left(1+te^{-s(k-1)}\right)^{-(k-1)}}{1+t\left(1+te^{-s(k-1)}\right)^{-(k-1)}}\right)\left(-\frac{t(k-1)e^{-s(k-1)}}{1+te^{-s(k-1)}}\right)\\ =& \frac{t^2(k-1)^2e^{-s(k-1)}}{\left(1+te^{-s(k-1)}\right)^k + t\left(1+te^{-s(k-1)}\right)}.\end{aligned}$$

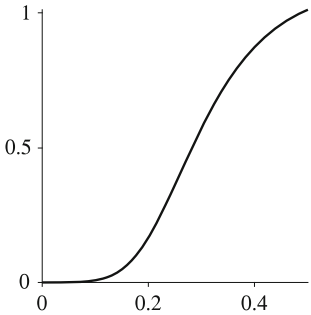


Fig. 6.5 The graph of $T(T(s))$ for $t = 2$ and $k = 11$

Thus we need to show that $(T(T(s))'$ is either decreasing or first increasing and then decreasing.

Equivalently, letting $y = e^{-s(k-1)}$, we have to show that the function

$$f(y) = \frac{(1+ty)^k + t(1+ty)}{y} \quad \text{for} \quad 0 \le y < 1$$

is either decreasing or first decreasing and then increasing. We write

$$f(y) = t(t+k) + \frac{1+t}{y} + \sum_{j=2}^{k} t^j \binom{k}{j} y^{j-1},$$

from which it follows that f is convex. Since

$$\lim_{y \longrightarrow 0+} f(y) = +\infty,$$

this proves that $f(y)$ is either decreasing for $0 < y \le 1$ or first decreasing and then increasing. Consequently, $T(T(s))'$ is either decreasing or first increasing and then decreasing. Therefore, $T(T(s))$ is either concave for $s \ge 0$ or has exactly one inflection point, where it changes from convex to concave.

Next, we observe that $s = a$, where a is the unique fixed point of T must also be a fixed point of T^2. If $b < a$ is a fixed point of T^2 then $c = T(b) > a$ is another fixed point of T^2 and if $c > a$ is a fixed point of T^2 then $b = T(c) < a$ is also a fixed point of T^2. Since T^2 has at most one inflection point, there cannot be more than 3 fixed points, see Fig. 6.6.

If there are three fixed points of T^2 then we must have $(T'(a))^2 = T(T(a))' > 1$ which means that $t > t_c$, see Fig. 6.6a. If $t < t_c$, we must have one fixed point of T^2, which is also the fixed point of T. Moreover, $T^2(s)$ is an increasing function such that $T^2(s) < s$ for $s < a$ and $T^2(s) > s$ for $s > a$.

It follows that if $t < t_c$ then for any $0 \le s < a$, the sequence $T^{2n}(s)$ is an increasing sequence converging to a, while for any $s > a$, the sequence is $T^{2n}(a)$ is a decreasing sequence converging to a, see Fig. 6.7.

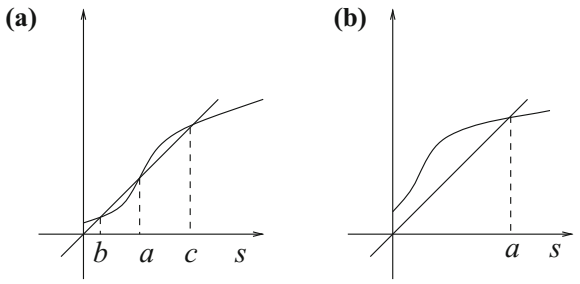


Fig. 6.6 The map T^2 has either three (**a**) or one (**b**) fixed points. In **a**, the fixed point a of T is locally repelling and in (**b**) it is locally attracting

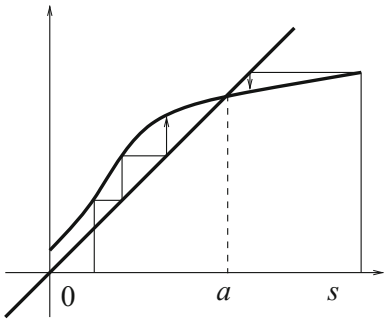


Fig. 6.7 Iterations $T^{2n}(s)$ for $s < a$ and $s > a$ when $t < t_c$

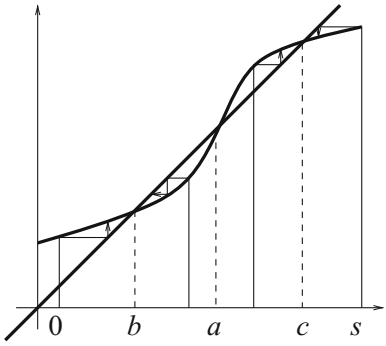


Fig. 6.8 Iterations $T^{2n}(s)$ when $t > t_c$.

Therefore,

$$\lim_{n\longrightarrow\infty} T^n(s) = a \quad \text{for all} \quad 0 \le s < \infty.$$

If $t > t_c$ then for any $0 \le s < b$ the sequence $T^{2n}(s)$ is an increasing sequence converging to b and for any $b < s < a$ the sequence $T^{2n}(s)$ is a decreasing sequence converging to b, while for any $a < s < c$ the sequence $T^{2n}(s)$ is an increasing sequence converging to c and for any $s > c$ the sequence $T^{2n}(s)$ is a decreasing sequence converging to c, see Fig. 6.8.

Therefore,

$$\lim_{n\longrightarrow\infty} T^{2n}(s) = b \quad \text{for all} \quad 0 \le s < a \qquad \text{and}$$
$$\lim_{n\longrightarrow\infty} T^{2n}(s) = c \quad \text{for all} \quad s > a.$$

It remains to show that if $t < t_c$ then $|T^n(s) - a|$ decreases exponentially fast with n. Since T switches sets $s < a$ and $s > a$ it suffices to prove exponential decay for one of the two sets. However, see Fig. 6.7, we have that

$$0 \le \frac{d}{ds}T^2(s) \le \frac{d}{ds}T^2(s)\Big|_{s=a} < \delta \quad \text{for all} \quad s \le a \quad \text{or for all} \quad s \ge a$$

and some $\delta = \delta(t) < 1$. Thus T^2 is a contraction for all $s > a$ or for all $s < a$, so that

$$\left|T^{2n}(a) - T^{2n}(s)\right| \le \delta^n |s - a| \quad \text{for all} \quad s > a \quad \text{or for all} \quad s < a.$$

Since for

$$a \le s \le +\infty \quad \text{we have} \quad T^2(a) = a \le T^2(s) \le T^2(+\infty) = \ln(1+t),$$

the proof follows. □

6.3.4 Correlation decay in trees $\mathbb{T}_n^k$. Let us consider the tree $\mathbb{T}_n^k$ of Sect. 6.3.2 and fix some subcritical

$$t < t_c = \frac{(k-1)^{k-1}}{(k-2)^k}.$$

Let $p_n^o = p_{k,n}^o(t)$ be the conditional probability that the root of $\mathbb{T}_n^k$ is occupied given that all vertices at the n-th level are occupied, see Fig. 6.9a.

Arguing as in Sects. 6.3.1 and 6.3.2, we conclude that p_n^o satisfies the recursion

$$1 - p_n^o = \frac{1}{1 + t\left(1 - p_{n-1}^o\right)^{k-1}} \quad \text{where} \quad p_0^o = 1.$$

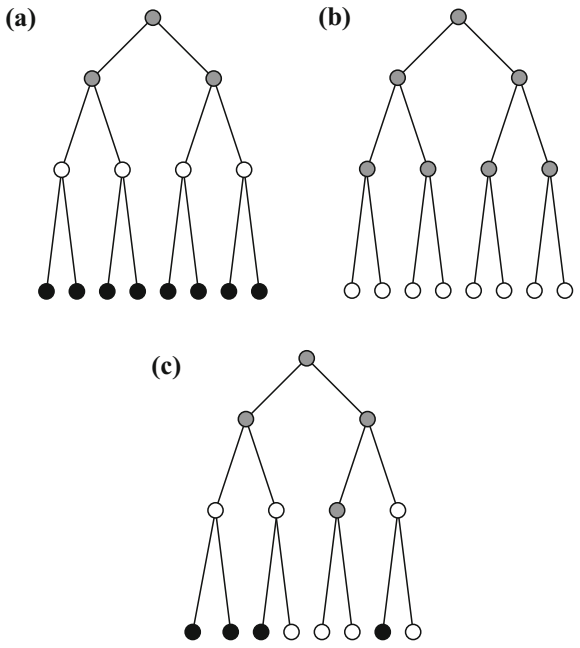


Fig. 6.9 *Black dots* are occupied vertices, *white dots* are unoccupied vertices and *grey dots* are the vertices that can be occupied or unoccupied

Hence by Theorem 6.3.3, we have

$$\lim_{n\longrightarrow\infty} p_n^o = 1 - x \tag{6.3.4.1}$$

where x is the unique real solution of the equation

$$x = \frac{1}{1 + tx^{k-1}}. \tag{6.3.4.2}$$

Moreover, the convergence in (6.3.4.1) is exponentially fast, meaning that

$$\left|\ln\left(1 - p_n^o\right) - \ln(1 - x)\right| \;\leq\; \gamma\delta^n,$$

for some $\gamma = \gamma(t, k) > 0$ and some $0 < \delta(t, k) < 1$.

Next, let $p_n^u = p_{k,n}^u(t)$ be the conditional probability that the root of $\mathbb{T}_n^k$ is occupied given that all vertices at the n-th level are unoccupied, see Fig. 6.9b. Arguing as in Sects. 6.3.1 and 6.3.2, we conclude that p_n^u satisfies the recursion

$$1 - p_n^u = \frac{1}{1 + t\left(1 - p_{n-1}^u\right)^{k-1}} \quad \text{where} \quad p_0^u = 0.$$

Hence by Theorem 6.3.3, we have

$$\lim_{n\longrightarrow\infty} p_n^u = 1 - x, \tag{6.3.4.3}$$

where x is the same unique real solution of the Eq. (6.3.4.2). Moreover, the convergence in (6.3.4.3) is exponentially fast, meaning that

$$\left|\ln\left(1 - p_n^u\right) - \ln(1 - x)\right| \;\leq\; \gamma\delta^n,$$

for some $\gamma = \gamma(t, k) > 0$ and some $0 < \delta(t, k) < 1$. In particular, the limits in (6.3.4.1) and (6.3.4.3) coincide.

Finally, let us impose impose some arbitrary occupancy constraints Λ at the n-th level of $\mathbb{T}_n^k$, see Fig. 6.9c and let $p_n^\Lambda = p_{n,k}^\Lambda(t)$ be the conditional probability that the root is occupied given those constraints. For a vertex v of $\mathbb{T}_n^k$, let $p^\Lambda(v)$ be the conditional probability that v is occupied given the constraints Λ at the n-th level. Arguing as in Sects. 6.3.1 and 6.3.2, we arrive to the recurrence

$$1 - p^\Lambda(v) = \frac{1}{1 + t\left(1 - p_{v_1}^\Lambda\right)\cdots\left(1 - p_{v_{k-1}}^\Lambda\right)}, \tag{6.3.4.4}$$

where $v_1, \ldots, v_{k-1}$ are the descendants of v and the initial conditions are $p^\Lambda(v) = 1$ if vertex v at the n-th level is occupied and $p^\Lambda(v) = 0$ if vertex v at the n-th level is unoccupied.

For vertices v at the n-th level, we clearly have

$$0 = p_0^u \;\leq\; p^\Lambda(v) \;\leq\; 1 = p_0^o.$$

From (6.3.4.4), for the vertices v at the $(n-1)$-st level, we have

$$0 = p_1^o \;\leq\; p^\Lambda(v) \;\leq\; p_1^u.$$

Iterating, we obtain,

$$p_m^u \;\leq\; p^\Lambda(v) \;\leq\; p_m^o,$$

when m is even and v is a vertex at the $(n-m)$-th level and

$$p_m^o \;\leq\; p^\Lambda(v) \;\leq\; p_m^u,$$

when m is odd and v is a vertex at the $(n-m)$-th level. Therefore,

$$\min\left\{p_n^u,\; p_n^o\right\} \;\leq\; p_n^\Lambda \;\leq\; \max\left\{p_n^u,\; p_n^o\right\}.$$

From (6.3.4.1) and (6.3.4.3), we conclude that

$$\lim_{n\longrightarrow\infty} p_n^\Lambda = 1 - x, \tag{6.3.4.5}$$

where x is the unique real solution of (6.3.4.2). In other words, asymptotically, as n grows, the conditional probability that the root is occupied does not depend on the occupancy constraint Λ at the n-th level if $\mathbb{T}_n^k$. Hence we say that for subcritical $t < t_c$ the model exhibits *correlation decay*. Moreover, the convergence in (6.3.4.5) is exponentially fast, meaning that

$$\left|\ln\left(1 - p_n^\Lambda\right) - \ln(1-x)\right| \;\leq\; \gamma\delta^n,$$

for some $\gamma = \gamma(t,k) > 0$ and some $0 < \delta = \delta(t,k) < 1$.

For supercritical values $t > t_c$ the root of the tree $\mathbb{T}_n^k$ remembers the occupancy constraint on the leaves, no matter how large n is. If $p_n^o = p_n^o(t)$ is the conditional probability that the root is occupied given that all leaves are occupied, we have

$$\lim_{n\longrightarrow\infty} p_{2n}^o \;>\; \lim_{n\longrightarrow\infty} p_{2n+1}^o$$

(both limits exist). Similarly, for the conditional probability $p_n^u = p_n^u(t)$ that the root is occupied given that all leaves are unoccupied, we have

$$\lim_{n\longrightarrow\infty} p_{2n}^u \;<\; \lim_{n\longrightarrow\infty} p_{2n+1}^u$$

(both limits exist). Imposing the condition that all leaves are occupied, makes the vertices on the m-th level more likely to be occupied if $m \equiv n \mod 2$ and less likely to be occupied if $m \equiv n+1 \mod 2$.

6.4 Correlation Decay for General Graphs

Our next goal is to show that the similar correlation decay for subcritical t holds not only for trees but also for general graphs. This was proved by Weitz [We06] and we follow his exposition. As the first and crucial step, we consider trees $\mathbb{T}_n^k$ with different non-negative real activities z_v at vertices.

6.4.1 Trees $\mathbb{T}_n^k$ with different activities at vertices. Suppose now that each vertex v of $\mathbb{T}_n^k$ has its own real activity $z_v \geq 0$. From (6.3.1.1), the probability $p(v)$ that vertex v is occupied satisfies

$$1 - p(v) = \frac{1}{1 + z_v\big(1 - p(u_1)\big) \cdots \big(1 - p(u_{k-1})\big)}, \tag{6.4.1.1}$$

where $u_1, \ldots, u_{k-1}$ are the descendants of v. Following [We06], we introduce ratios

$$r(v) = \frac{p(v)}{1 - p(v)}$$

for each vertex v of $\mathbb{T}_n^k$. Then

$$0 \leq r(v) \leq +\infty, \quad p(v) = \frac{r(v)}{1 + r(v)}$$

and the recursion (6.4.1.1) is written as

$$r(v) = \frac{z_v}{\big(1 + r(u_1)\big) \cdots \big(1 + r(u_{k-1})\big)}, \tag{6.4.1.2}$$

where $u_1, \ldots, u_{k-1}$ are the descendants of v.

Let $r_n^{\max} = r_n^{\max}(z)$ denote the largest possible value of $r(v)$ at the root v of $\mathbb{T}_n^k$ given the vector of activities $z = (z_u : u \in V)$ where the maximum is taken over all possible choices of the initial values $r(u)$ at the leaves u of $\mathbb{T}_n^k$ and let $r_n^{\min} = r_n^{\min}(z)$ denote the smallest possible value of $r(v)$ at the root v of $\mathbb{T}_n^k$ given the vector of activities $z = (z_u : u \in V)$ where the minimum is taken over all possible choices of the initial values $r(u)$ at the leaves u of $\mathbb{T}_n^k$. We denote $r_n^{\max}(t)$, respectively $r_n^{\min}(t)$ the corresponding quantities when $z_u = t$ for some $t \geq 0$ and all vertices u.

6.4.2 Theorem. *Suppose that*

$$0 < z_u \leq t$$

for some $t > 0$ *and all vertices* u *of* $\mathbb{T}_n^k$. *Then for* $n \geq 2$, *we have* $r_n^{\max}(z) < +\infty$, $r_n^{\min}(z) > 0$ *and the inequalities*

$$\frac{r_n^{\max}(z)}{r_n^{\min}(z)} \leq \frac{r_n^{\max}(t)}{r_n^{\min}(t)} \tag{6.4.2.1}$$

and

$$\frac{1 + r_n^{\max}(z)}{1 + r_n^{\min}(z)} \leq \frac{1 + r_n^{\max}(t)}{1 + r_n^{\min}(t)} \tag{6.4.2.2}$$

hold. In addition, $r_1^{\max}(z) < +\infty$ *and (6.4.2.2) holds for* $n = 1$.

Some remarks are in order. As follows from (6.4.1.2), if n is odd, the value of $r_n^{\max}$ is attained when $r(v) = 0$ for all leaves v of $\mathbb{T}_n^k$ and the value of $r_n^{\min}$ is attained when $r(v) = +\infty$ for all leaves v of $\mathbb{T}_n^k$, while if n is even, the value of $r_n^{\max}$ is attained when $r(v) = +\infty$ for all leaves v of $\mathbb{T}_n^k$ and the value of $r_n^{\min}$ is attained when $r(v) = 0$ for all leaves v of $\mathbb{T}_n^k$. By continuity, inequality (6.4.2.2) holds when $0 \leq z_v \leq t$. It can be written as

$$\frac{1 - p_n^{\min}(z)}{1 - p_n^{\max}(z)} \leq \frac{1 - p_n^{\min}(t)}{1 - p_n^{\max}(t)},$$

where $p^{\max}$, respectively, $p^{\min}$ is the maximum, respectively minimum, probability that the root is occupied taken over all possible initial occupancy probabilities $0 \leq p(v) \leq 1$ on the leaves of $\mathbb{T}_n^k$. As is discussed in Sect. 6.3.4, for subcritical values

$$t < t_c = \frac{(k-1)^{k-1}}{(k-2)^k}$$

we have

$$\lim_{n \longrightarrow \infty} \frac{1 - p_n^{\min}(t)}{1 - p_n^{\max}(t)} = 1$$

and hence necessarily

$$\lim_{n \longrightarrow \infty} \frac{1 - p_n^{\min}(z)}{1 - p_n^{\max}(z)} = 1.$$

In other words, the tree $\mathbb{T}_n^k$ with different subcritical activities at each vertex also exhibits correlation decay.

We prove Theorem 6.4.2 by induction on n. First, we establish some inequalities.

6.4.3 Lemma.

(1) Let a, b, c and d be non-negative numbers such that $b > 0$, $d > 0$,

$$a \le c \quad \text{and} \quad 1 \le \frac{a}{b} \le \frac{c}{d}.$$

Then

$$\frac{1+a}{1+b} \le \frac{1+c}{1+d}.$$

(2) Let $c \ge b \ge 0$ be reals. Then for any $\alpha \ge \delta \ge 0$, we have

$$\frac{1+\alpha c}{1+\alpha b} \ge \frac{1+\delta c}{1+\delta b}.$$

Proof. We have

$$\frac{1+c}{1+d} - \frac{1+a}{1+b} = \frac{(1+c)(1+b)-(1+a)(1+d)}{(1+d)(1+b)} = \frac{(b+c)-(a+d)+(cb-ad)}{(1+d)(1+b)}.$$

Since

$$\frac{c}{d} - \frac{a}{b} = \frac{cb-ad}{db} \ge 0,$$

we conclude that $cb - ad \ge 0$.

Writing $c = \gamma a$ for some $\gamma \ge 1$, we conclude that $d \le \gamma b$ and hence

$$(b+c)-(a+d) \ge b + \gamma a - a - \gamma b = (\gamma - 1)(a-b) \ge 0$$

and the proof of Part (1) follows.

To prove Part (2), we note that

$$\frac{1+\alpha c}{1+\alpha b} - \frac{1+\delta c}{1+\delta b} = \frac{(1+\alpha c)(1+\delta b)-(1+\delta c)(1+\alpha b)}{(1+\delta b)(1+\alpha b)} = \frac{(c-b)(\alpha-\delta)}{(1+\delta b)(1+\alpha b)} \ge 0.$$

□

6.4.4 Lemma. *Let $k > 2$ be a positive integer, and t, b and c be non-negative real such that*

$$b \le c, \quad c \ge \frac{t}{(1+b)^{k-1}} \quad \text{and} \quad b \le \frac{t}{(1+c)^{k-1}}.$$

Let us define a function

$$f(\alpha_1, \ldots, \alpha_{k-1}) = \frac{1 + t(1+\alpha_1 b)^{-1} \cdots (1+\alpha_{k-1} b)^{-1}}{1 + t(1+\alpha_1 c)^{-1} \cdots (1+\alpha_{k-1} c)^{-1}} \quad \text{for} \quad \alpha_1, \ldots, \alpha_{k-1} \ge 0.$$

Then

$$f(\alpha_1, \ldots, \alpha_{k-1}) \leq f(1, \ldots, 1) \quad \textit{for all} \quad 0 \leq \alpha_1, \ldots, \alpha_{k-1} \leq 1.$$

Proof. We need to prove that

$$\begin{aligned} & f(1, \ldots, 1)\left(1 + t\,(1+\alpha_1 c)^{-1} \cdots (1+\alpha_{k-1} c)^{-1}\right) \\ & \quad \geq 1 + t\,(1+\alpha_1 b)^{-1} \cdots (1+\alpha_{k-1} b)^{-1} \end{aligned}$$

provided $0 \leq \alpha_1, \ldots, \alpha_{k-1} \leq 1$. Since for $\alpha_1 = \ldots = \alpha_{k-1} = 1$ we attain equality above, it suffices to prove that the function

$$\begin{aligned} g(\alpha_1, \ldots, \alpha_{k-1}) = & 1 + t\,(1+\alpha_1 b)^{-1} \cdots (1+\alpha_{k-1} b)^{-1} \\ & - f(1, \ldots, 1) - t f(1, \ldots, 1)\,(1+\alpha_1 c)^{-1} \cdots (1+\alpha_{k-1} c)^{-1} \end{aligned}$$

is non-decreasing in every variable $0 \leq \alpha_i \leq 1$ provided the remaining variables $0 \leq \alpha_j \leq 1$ are fixed. By symmetry, it suffices to check that

$$\frac{\partial}{\partial \alpha_1} g(\alpha_1, \ldots, \alpha_{k-1}) \geq 0$$

provided $0 \leq \alpha_1, \ldots, \alpha_{k-1} \leq 1$. Computing the derivative, we obtain

$$\begin{aligned} \frac{\partial}{\partial \alpha_1} g(\alpha_1, \ldots, \alpha_{k-1}) = & -tb\,(1+\alpha_1 b)^{-1}\,(1+\alpha_1 b)^{-1} \cdots (1+\alpha_{k-1} b)^{-1} \\ & + tc f(1, \ldots, 1)\,(1+\alpha_1 c)^{-1}\,(1+\alpha_1 c)^{-1} \cdots (1+\alpha_{k-1} c)^{-1}. \end{aligned}$$

Hence it suffices to prove that

$$\frac{1+\alpha_1 c}{1+\alpha_1 b} \cdot \frac{(1+\alpha_1 c) \cdots (1+\alpha_{k-1} c)}{(1+\alpha_1 b) \cdots (1+\alpha_{k-1} b)} \leq \frac{c}{b} f(1, \ldots, 1). \tag{6.4.4.1}$$

On the other hand,

$$\begin{aligned} \frac{c}{b} f(1, \ldots, 1) = & \frac{c + tc(1+b)^{-(k-1)}}{b + tb(1+c)^{-(k-1)}} \geq \frac{t(1+b)^{-(k-1)} + tc(1+b)^{-(k-1)}}{t(1+c)^{-(k-1)} + tb(1+c)^{-(k-1)}} \\ = & \frac{(1+c)(1+b)^{-(k-1)}}{(1+b)(1+c)^{-(k-1)}} = \frac{(1+c)^k}{(1+b)^k}. \end{aligned} \tag{6.4.4.2}$$

By Part (2) of Lemma 6.4.3, we have

$$\frac{1+\alpha_i c}{1+\alpha_i b} \leq \frac{1+c}{1+b} \quad \text{for} \quad i = 1, \ldots, k-1$$

and hence

$$\frac{1+\alpha_1 c}{1+\alpha_1 b}\cdot\frac{(1+\alpha_1 c)\cdots(1+\alpha_{k-1}c)}{(1+\alpha_1 b)\cdots(1+\alpha_{k-1}b)} \;\le\; \frac{(1+c)^k}{(1+b)^k}. \tag{6.4.4.3}$$

Combining (6.4.4.3) and (6.4.4.2), we obtain (6.4.4.1) and hence complete the proof. □

The final lemma before we embark on the proof of Theorem 6.4.2.

6.4.5 Lemma. *For any $t > 0$ we have*

$$r_n^{\max}(t) \;\le\; r_{n-1}^{\max}(t) \quad \textit{and} \quad r_n^{\min}(t) \;\ge\; r_{n-1}^{\min}(t)$$

for all positive integer n.

Proof. We proceed by induction on n. We have

$$r_0^{\max}(t) = +\infty \quad \text{and} \quad r_0^{\min}(t) = 0$$

and, by (6.4.1.2),

$$r_1^{\max}(t) = \frac{t}{\left(1+r_0^{\min}(t)\right)^{k-1}} = t \;<\; r_0^{\max}(t)$$

and

$$r_1^{\min}(t) = \frac{t}{\left(1+r_0^{\max}(t)\right)^{k-1}} = 0 \;=\; r_0^{\min}(t).$$

For $n > 1$ by (6.4.1.2) and the induction hypothesis, we have

$$r_n^{\max}(t) = \frac{t}{\left(1+r_{n-1}^{\min}(t)\right)^{k-1}} \;\le\; \frac{t}{\left(1+r_{n-2}^{\min}(t)\right)^{k-1}} = r_{n-1}^{\max}(t)$$

and, similarly,

$$r_n^{\min}(t) = \frac{t}{\left(1+r_{n-1}^{\max}(t)\right)^{k-1}} \;\ge\; \frac{t}{\left(1+r_{n-2}^{\max}(t)\right)^{k-1}} = r_{n-1}^{\min}(t),$$

which completes the proof. □

6.4.6 Proof of Theorem 6.4.2. Let v be the root of $\mathbb{T}_n^k$. We have

$$r_0^{\max}(z) = r_0^{\max}(t) = +\infty \quad \text{and} \quad r_0^{\min}(z) = r_0^{\min}(t) = 0,$$

from which

$$r_1^{\max}(z) = z_v, \quad r_1^{\max}(t) = t, \quad r_1^{\min}(z) = r_1^{\min}(t) = 0.$$

Since $z_v \le t$, we have

$$\frac{1 + r_1^{\max}(z)}{1 + r_1^{\min}(z)} = 1 + z_v \ \le \ 1 + t = \frac{1 + r_1^{\max}(t)}{1 + r_1^{\min}(t)}$$

which proves (6.4.4.2) for $n = 1$.

If we remove v with adjacent edges from $\mathbb{T}_n^k$, we obtain a vertex-disjoint union of $k-1$ trees $\mathbb{T}_{n-1}^k$, the i-th tree with activity vector z_i satisfying $z_i(u) \le t$ for all u. Applying (6.4.1.2), we obtain

$$\begin{aligned} r_n^{\max}(z) &= \frac{z_v}{\big(1 + r_{n-1}^{\min}(z_1)\big) \cdots \big(1 + r_{n-1}^{\min}(z_{k-1})\big)} \quad \text{and} \\ r_n^{\min}(z) &= \frac{z_v}{\big(1 + r_{n-1}^{\max}(z_1)\big) \cdots \big(1 + r_{n-1}^{\max}(z_{k-1})\big)} \end{aligned} \tag{6.4.6.1}$$

and, similarly,

$$r_n^{\max}(t) = \frac{t}{\big(1 + r_{n-1}^{\min}(t)\big)^{k-1}} \quad \text{and} \quad r_n^{\min}(t) = \frac{t}{\big(1 + r_{n-1}^{\max}(t)\big)^{k-1}}.$$

We proceed by induction on n. For $n = 2$, by (6.4.6.1) we have

$$r_2^{\max}(z) = z_v, \quad r_2^{\max}(t) = t, \quad r_2^{\min}(z) > 0, \quad r_2^{\min}(t) > 0$$

and

$$\frac{r_2^{\max}(z)}{r_2^{\min}(z)} = \frac{1 + r_1^{\max}(z_1)}{1 + r_1^{\min}(z_1)} \cdots \frac{1 + r_1^{\max}(z_{k-1})}{1 + r_1^{\min}(z_{k-1})} \ \le \ \left(\frac{1 + r_1^{\max}(t)}{1 + r_1^{\min}(t)}\right)^{k-1} = \frac{r_2^{\max}(t)}{r_2^{\min}(t)},$$

which establishes (6.4.2.1) for $n = 2$. Moreover, since $r_2^{\max}(z) \le r_2^{\max}(t)$, the inequality (4.3.2) follows by Part (1) of Lemma 6.4.3.

Suppose that $n > 2$. Applying the induction hypothesis, we obtain from (6.4.6.1)

$$\frac{r_n^{\max}(z)}{r_n^{\min}(z)} = \frac{1 + r_{n-1}^{\max}(z_1)}{1 + r_{n-1}^{\min}(z_1)} \cdots \frac{1 + r_{n-1}^{\max}(z_{k-1})}{1 + r_{n-1}^{\min}(z_{k-1})} \ \le \ \left(\frac{1 + r_{n-1}^{\max}(t)}{1 + r_{n-1}^{\min}(t)}\right)^{k-1} = \frac{r_n^{\max}(t)}{r_n^{\min}(t)}.$$

In particular, $r_n^{\max}(z) < +\infty$, $r_n^{\min}(z) > 0$ and (6.4.2.1) follows.

Hence our goal is to prove (6.4.2.2).

First, we observe that if

$$r_n^{\max}(z) \ \le \ r_n^{\max}(t)$$

then (6.4.2.2) follows by (6.4.2.1) and Part (1) of Lemma 6.4.3. Hence without loss of generality, we may assume that

$$r_n^{\max}(z) \;>\; r_n^{\max}(t). \tag{6.4.6.2}$$

Let z' be the vector of activities obtained from z by replacing the activity z_v of the root by $t \geq z_v$. From (6.4.6.1) it follows that

$$r_n^{\max}(z') = \frac{t}{z_v} r_n^{\max}(z) \quad \text{and} \quad r_n^{\min}(z') = \frac{t}{z_v} r_n^{\min}(z)$$

so that by Part (2) of Lemma 6.4.3, we have

$$\frac{1 + r_n^{\max}(z')}{1 + r_n^{\min}(z')} \;\geq\; \frac{1 + r_n^{\max}(z)}{1 + r_n^{\min}(z)}.$$

Therefore, without loss of generality, we may assume that

$$z_v = t. \tag{6.4.6.3}$$

Recall that z_i is the vector of activities at the vertices of the i-th tree $\mathbb{T}_{n-1}^k$ obtained from $\mathbb{T}_n^k$ by removing the root v with the adjacent edges. Let $I \subset \{1, \ldots, k-1\}$ be the set of indices i such that

$$r_{n-1}^{\min}(z_i) \;\geq\; r_{n-1}^{\min}(t).$$

Let z' be the vector of activities at the vertices of $\mathbb{T}_n^k$ obtained by replacing the vector z_i of activities at the vertices of the i-th tree $\mathbb{T}_{n-1}^k$ by t for all $i \in I$ and let z_i' be the corresponding vector of activities at the vertices of the i-th tree $\mathbb{T}_{n-1}^k$ (hence $z_i' = z_i$ if $i \notin I$ and z_i' is the constant vector of t if $i \in I$).

By the induction hypothesis,

$$\frac{1 + r_{n-1}^{\max}(z_i)}{1 + r_{n-1}^{\min}(z_i)} \;\leq\; \frac{1 + r_{n-1}^{\max}(t)}{1 + r_{n-1}^{\min}(t)}$$

and using (6.4.6.1) and (6.4.6.3), we conclude that

$$\frac{r_n^{\max}(z)}{r_n^{\min}(z)} \;\leq\; \frac{r_n^{\max}(z')}{r_n^{\min}(z')}.$$

Moreover, we have

$$r_{n-1}^{\min}(z_i') \;\leq\; r_{n-1}^{\min}(z_i) \quad \text{for all} \quad i = 1, \ldots, k-1$$

and by (6.4.6.1) and (6.4.6.3) we have

$$r_n^{\max}(z) \;\le\; r_n^{\max}(z').$$

It follows then by Part (1) of Lemma 6.4.3 that

$$\frac{1+r_n^{\max}(z)}{1+r_n^{\min}(z)} \;\le\; \frac{1+r_n^{\max}(z')}{1+r_n^{\min}(z')}.$$

Therefore, without loss of generality, we may assume that $I=\emptyset$ and hence

$$r_{n-1}^{\min}(z_i) \;\le\; r_{n-1}^{\min}(t) \quad \text{for} \quad i=1,\ldots,k-1. \tag{6.4.6.4}$$

In view of (6.4.6.4), let us define $0 \le \alpha_1,\ldots,\alpha_{k-1} \le 1$ such that

$$r_{n-1}^{\min}(z_i) = \alpha_i r_{n-1}^{\min}(t) \quad \text{for} \quad i=1,\ldots,k-1.$$

By the induction hypothesis,

$$\frac{r_{n-1}^{\max}(z_i)}{r_{n-1}^{\min}(z_i)} \;\le\; \frac{r_{n-1}^{\max}(t)}{r_{n-1}^{\min}(t)}$$

and hence

$$r_{n-1}^{\max}(z_i) \;\le\; \alpha_i r_{n-1}^{\max}(t) \quad \text{for} \quad i=1,\ldots,k-1.$$

Applying (6.4.6.1) and (6.4.6.3), we conclude that

$$\frac{1+r_n^{\max}(z)}{1+r_n^{\min}(z)} \;\le\; \frac{1+t\left(1+\alpha_1 r_{n-1}^{\min}(t)\right)^{-1}\cdots\left(1+\alpha_{k-1}r_{n-1}^{\min}(t)\right)^{-1}}{1+t\left(1+\alpha_1 r_{n-1}^{\max}(t)\right)^{-1}\cdots\left(1+\alpha_{k-1}r_{n-1}^{\min}(t)\right)^{-1}}.$$

Besides, from Lemma 6.4.5,

$$r_{n-1}^{\max}(t) \;\ge\; r_n^{\max}(t) = \frac{t}{\left(1+r_{n-1}^{\min}(t)\right)^{k-1}} \quad \text{and}$$

$$r_{n-1}^{\min}(t) \;\le\; r_n^{\min}(t) = \frac{t}{\left(1+r_{n-1}^{\max}(t)\right)^{k-1}}.$$

Applying Lemma 6.4.4, we obtain

$$\frac{1+r_n^{\max}(z)}{1+r_n^{\min}(z)} \;\le\; \frac{1+t\left(1+r_{n-1}^{\min}(t)\right)^{-(k-1)}}{1+t\left(1+r_{n-1}^{\max}(t)\right)^{-(k-1)}} = \frac{1+r_n^{\max}(t)}{1+r_n^{\min}(t)}$$

which proves (6.4.2.2) and completes the proof of the theorem. □

6.4.7 Correlation decay for general graphs. Let $G = (V, E)$ be a general graph and suppose that the degrees of vertices do not exceed $\Delta \geq 3$. Weitz [We06] showed that if

$$t < t_c = \frac{(\Delta - 1)^{\Delta - 1}}{(\Delta - 2)^{\Delta}}$$

then the probability $p(v)$ that a particular vertex is occupied is asymptotically independent on whether vertices far away from v are occupied (as in Sect. 5.2.3, we measure the distance between a pair of vertices by the smallest number of edges in a path connecting the vertices). Weitz [We06] deduced this result from Theorem 6.4.2, and we sketch the reduction here.

First, we note that Theorem 6.4.2 implies correlation decay on k-regular trees with subcritical activities

$$0 \leq z_u \leq t < t_c = \frac{(k-1)^{k-1}}{(k-2)^k} \tag{6.4.7.1}$$

at the vertices. Indeed, suppose that v is the root of a k-regular tree with n levels, see Fig. 5.3, and let $u_1, \dots, u_k$ be the neighbors of v. Let us impose some occupancy condition Λ on the leaves of the tree (that is, set some leaves as occupied, as the rest as unoccupied). If we remove v with incident edges, the remaining graph splits into the vertex-disjoint union of k trees $\mathbb{T}^k_{n-1}$, and from (6.3.1.1) we deduce the following recursive relation

$$1 - p^{\Lambda}(v) = \frac{1}{1 + z_v \left(1 - p^{\Lambda}(u_1)\right) \cdots \left(1 - p^{\Lambda}(u_k)\right)}$$

for the probabilities $p^{\Lambda}(u)$ of occupancy. Theorem 6.4.2 implies that as n grows, the probabilities $p^{\Lambda}(u_1), \dots, p^{\Lambda}(u_k)$ converge to limits independent on the occupancy condition Λ at the leaves of the tree and hence the probability $p^{\Lambda}(v)$ that the root is occupied also converges to a limit independent of Λ.

The next observation is that we have correlation decay if G is a tree where the degree of every vertex is at most k and subcritical activities (6.4.7.1) at every vertex. This case reduces to the case of a k-regular tree by adding auxiliary vertices where needed with zero activities, cf. Fig. 6.10.

Finally, Weitz [We06] reduces the case of a general graph $G = (V, E)$ with largest degree $\Delta(G) > 2$ of a vertex and subcritical activities at the vertices to the case of a tree with degrees of the vertices not exceeding $\Delta(G)$. We present a modification of that construction suggested by Gamarnik [Ga16].

We start by rewriting (6.3.1.1) for the case when $z_v = t$ for all $v \in V$:

$$\frac{\mathrm{ind}_{G-v}(t)}{\mathrm{ind}_G(t)} = \frac{1}{1 + t\frac{\mathrm{ind}_{G-v-N_v}(t)}{\mathrm{ind}_{G-v}(t)}}. \tag{6.4.7.2}$$

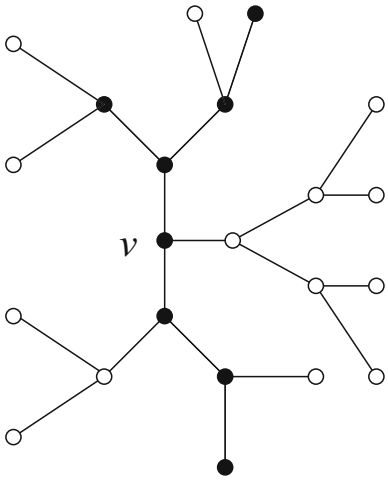


Fig. 6.10 A tree (*black* nodes) with a vertex v appended (*white* nodes) to a 3-regular tree with root v

Let $v_1, \ldots, v_k$, $k \leq \Delta(G)$, be the vertices of N_v (that is, the neighbors of v), listed in some order. We can further rewrite

$$\begin{aligned}\frac{\mathrm{ind}_{G-v-N_v}(t)}{\mathrm{ind}_{G-v}(t)} =& \frac{\mathrm{ind}_{G-v-v_1}(t)}{\mathrm{ind}_{G-v}(t)} \cdot \frac{\mathrm{ind}_{G-v-v_1-v_2}(t)}{\mathrm{ind}_{G-v-v_1}(t)} \cdots \\ & \times \frac{\mathrm{ind}_{G-v-v_1-\ldots-v_k}(t)}{\mathrm{ind}_{G-v-v_1-\ldots-v_{k-1}}(t)}.\end{aligned} \tag{6.4.7.3}$$

Let $p(v, v_1, \ldots, v_i; v_{i+1})$ be the conditional probability that a random independent set contains v_{i+1} given that it does not contain any of the vertices $v, v_1, \ldots, v_i$. Then

$$1 - p\left(v, v_1 \ldots, v_i; v_{i+1}\right) = \frac{\mathrm{ind}_{G-v-v_1-\ldots-v_{i+1}}(t)}{\mathrm{ind}_{G-v-v_1-\ldots-v_i}(t)}$$

and combining (6.4.7.2) and (6.4.7.3), we obtain

$$1 - p(v) = \frac{1}{1 + t\left(1 - p\left(v; v_1\right)\right)\left(1 - p\left(v, v_1; v_2\right)\right) \cdots \left(1 - p\left(v, v_1, \ldots, v_{k-1}; v_k\right)\right)}.$$

On the other hand, each of the probabilities $p\left(v, v_1, \ldots, v_i; v_{i+1}\right)$ can be computed as the probability of occupancy of v_{i+1} in the graph $G - v - v_1 - \ldots - v_i$ obtained from G by removing the vertices $v, v_1, \ldots, v_i$ together with incident edges. This allows us to arrange the computation of $p(v)$ recursively into a tree. For example, for suppose we want to compute the probability $p(v)$ of occupancy in the graph on Fig. 6.11.

Then we obtain the tree pictured on Fig. 6.12.

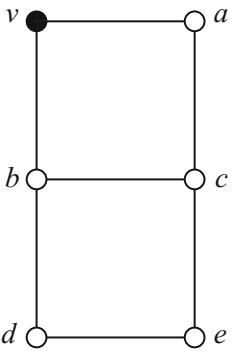


Fig. 6.11 A graph and a vertex v

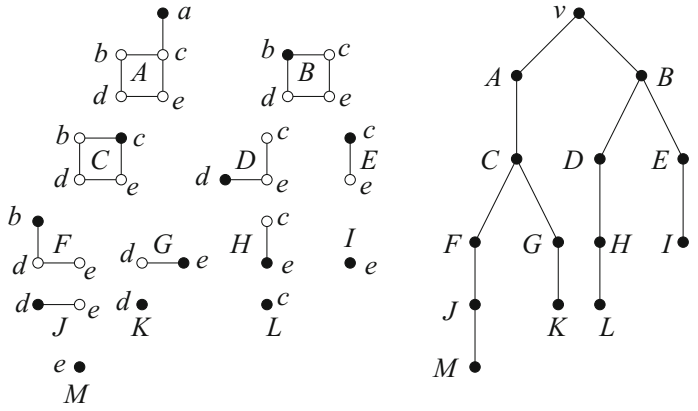


Fig. 6.12 Computational tree to compute the occupancy probability $p(v)$ for the graph on Fig. 6.11. We recursively compute occupancy probabilities for *black* nodes in the corresponding subgraphs of the graph

Denoting by $p_X(u)$ the occupancy probability of a vertex u in a graph X, we obtain recursively:

$$1 - p_M(e) = 1 - p_K(d) = 1 - p_L(c) = 1 - p_I(e) = \frac{1}{1+t},$$

$$1 - p_J(d) = 1 - p_G(e) = 1 - p_H(e) = 1 - p_E(c) = \frac{1}{1+t\frac{1}{1+t}} = \frac{1+t}{1+2t},$$

$$1 - p_F(b) = 1 - p_D(d) = \frac{1}{1+t\frac{1+t}{1+2t}} = \frac{1+2t}{1+3t+t^2},$$

$$1 - p_C(c) = 1 - p_B(b) = \frac{1}{1+t\frac{1+2t}{1+3t+t^2}\frac{1+t}{1+2t}} = \frac{1+3t+t^2}{1+4t+2t^2},$$

$$1 - p_A(a) = \frac{1}{1 + t\frac{1+3t+t^2}{1+4t+2t^2}} = \frac{1+4t+2t^2}{1+5t+5t^2+t^3} \quad \text{and}$$

$$1 - p(v) = \frac{1}{1 + t\frac{1+4t+2t^2}{1+5t+5t^2+t^3}\frac{1+3t+t^2}{1+4t+2t^2}} = \frac{1+5t+5t^2+5t^3}{1+6t+8t^2+2t^3}$$

so that finally

$$p(v) = \frac{t+3t^2+t^3}{1+6t+8t^2+2t^3}.$$

Indeed, it is easy to see that for the graph on Fig. 6.11, there are two independent sets of 3 vertices, one of which contains v, there are 8 independent sets of 2 vertices, three of which contain v, there are 6 independent sets of one vertex, one of which contains v, there is a unique independent set of 0 vertices not containing v and there are no independent sets of 4 or more vertices.

This construction establishes correlation decay for general graphs of maximum degree k and subcritical activities z_u satisfying (6.4.7.1). Using telescoping as in Sect. 5.2.3, Weitz [We06] further deduced that for such a family of graphs one can approximate $\mathrm{ind}_G(z)$ for non-negative weights $z = (z_v)$ within relative error ϵ in time polynomial in $|V|$ and ϵ^{-1} as long as (6.4.7.1) holds. In particular, as long as $\Delta(G) \le 5$, the value of $\mathrm{ind}_G(1, \ldots, 1)$, that is, the number of independent sets in G, can be efficiently approximated. On the other hand, Sly [Sl10] and Sly and Sun [SS14] showed that the approximate counting of independent sets in computationally hard when (6.4.7.1) is violated.

6.5 The Roots on and Near the Real Axis

We note that

$$\frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}} = \frac{e}{\Delta}\left(1 + O\left(\frac{1}{\Delta}\right)\right) \quad \text{as} \quad \Delta \longrightarrow +\infty.$$

Although the above bound and (6.1.5.1) are both inversely proportional to $\Delta(G)$, the correlation decay bound above achieves a better constant.

Sokal conjectured [S01b] that for any $0 < \epsilon < 1$ there exists $\delta = \delta(\epsilon) > 0$ such that for any graph G with the largest degree of a vertex not exceeding $\Delta > 2$, we have $\mathrm{ind}_G(z, \ldots, z) \neq 0$ provided

$$0 \le \Re\, z \ \le\ (1-\epsilon)\frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}} \quad \text{and} \quad |\Im\, z| \ \le\ \delta.$$

Should this conjecture be true, the technique of Lemma 1.2.3, see also [PR16] and Sections 3.7, 5.1.7 and 6.1.5, would allow us to bridge the gap between the

approximations achievable via the Taylor polynomial method and the correlation decay method[1].

Below we present a result of Regts [Re16] confirming the absence of the roots near the positive real axis "halfway between" the Dobrushin - Scott - Sokal bound (6.1.5.1) and the conjectured Sokal bound.

6.5.1 Theorem. *Let us choose an* $0 < \epsilon < 1$. *Let* G *be a graph with the largest degree of vertex not exceeding* $\Delta \geq 2$. *Then*

$$\mathrm{ind}_G(z) \neq 0$$

for all activities $z = (z_v)$ *such that*

$$|z_v| \leq \tan \frac{\pi}{(2+2\epsilon)(\Delta-1)} \quad \textit{and} \quad |\arg z_v| \leq \frac{\epsilon\pi}{2+2\epsilon} \quad \textit{for all} \quad v \in V.$$

The proof is based on the following geometric lemma.

6.5.2 Lemma. *Let us fix a real* $0 < \epsilon < 1$, *let* $d \geq 1$ *be an integer and for* $k \leq d$ *let* $w_1, \ldots, w_k$ *be complex numbers such that*

$$\left|w_j\right| \leq 1 \quad \textit{and} \quad \left|\arg w_j\right| \leq \frac{\pi}{(2+2\epsilon)d} \quad \textit{for} \quad j = 1, \ldots, k.$$

Let z *be a complex number such that*

$$|z| \leq \tan \frac{\pi}{(2+2\epsilon)d} \quad \textit{and} \quad |\arg z| \leq \frac{\epsilon\pi}{2+2\epsilon}$$

and let

$$w = \frac{1}{1 + z w_1 \cdots w_k}$$

Then

$$|w| \leq 1 \quad \textit{and} \quad |\arg w| \leq \frac{\pi}{(2+2\epsilon)d}.$$

Proof. Clearly,

$$|w_1 \cdots w_k| \leq 1 \quad \text{and} \quad |\arg w_1 \cdots w_k| \leq \frac{\pi}{2+2\epsilon}.$$

In particular, $\Re\,(z w_1 \cdots w_k) \geq 0$ and hence $|1 + z w_1 \ldots w_k| \geq 1$ and $|w| \leq 1$. Moreover, see Fig. 6.13,

$$|\arg\,(1 + z w_1 \ldots w_k)| \leq \arctan |z w_1 \cdots w_k| \leq \arctan |z| \leq \frac{\pi}{(2+2\epsilon)d}.$$

[1] Added in Proofs: The conjecture was proved in H. Peters and G. Regts, "On a conjecture of Sokal concerning roots of the independence polynomial", preprint arXiv:1701.08049 (2017)

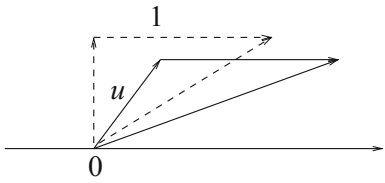


Fig. 6.13 The real axis (horizontal), the vectors $u = zw_1 \cdots w_k$ and $1 + u$

The proof now follows. □

6.5.3 Proof of Theorem 6.5.1. The proof is somewhat similar to that of Theorem 6.1.3. We proceed by induction on the number $|V|$ of vertices of G (the outer induction). If $|V| = 1$, the result clearly holds, so we assume that $|V| > 1$.

We embed in the proof another inductive argument. Namely, we prove by induction on $|V|$ that if $G = (V, E)$ is a graph of the largest degree $\Delta(G) \le \Delta$ of a vertex and if v is a vertex of degree at most $\Delta - 1$ then

$$\begin{aligned} &\mathrm{ind}_G(z) \neq 0, \quad \left|\frac{\mathrm{ind}_{G-v}(z)}{\mathrm{ind}_G(z)}\right| \le 1 \quad \text{and} \\ &\left|\arg \frac{\mathrm{ind}_{G-v}(z)}{\mathrm{ind}_G(z)}\right| \le \frac{\pi}{(2+2\epsilon)(\Delta-1)}. \end{aligned} \tag{6.5.3.1}$$

The case of $|V| = 1$ is easy to check, so we assume that $|V| \ge 2$. As in the proof of Theorem 6.1.3, we use the recursive formulas (6.1.3.1) and (6.1.3.2) and note that the product in the right hand side of (6.1.3.2) contains $k \le \Delta - 1$ factors. If $k = 0$, so that v is an isolated vertex of G then $\mathrm{ind}_{G-v}(z) \neq 0$ by the outer induction hypothesis and

$$\frac{\mathrm{ind}_{G-v}(z)}{\mathrm{ind}_G(z)} = \frac{1}{1+z_v},$$

so that (6.5.3.1) holds. Hence we assume that $k > 0$ and v has neighbors $v_1, \ldots, v_k$ in G.

Since the degree of v_i in $G - v - v_1 - \ldots - v_{i-1}$ does not exceed $\Delta - 1$, by the induction hypothesis, we have

$$\begin{aligned} &\left|\frac{\mathrm{ind}_{G-v-v_1-\ldots-v_{i-1}}(z)}{\mathrm{ind}_{G-v-v_1-\ldots-v_i}(z)}\right| \le 1 \quad \text{and} \\ &\left|\arg \frac{\mathrm{ind}_{G-v-v_1-\ldots-v_{i-1}}(z)}{\mathrm{ind}_{G-v-v_1-\ldots-v_i}(z)}\right| \le \frac{\pi}{(2+2\epsilon)(\Delta-1)} \quad \text{for} \quad i = 1, \ldots, k. \end{aligned} \tag{6.5.3.2}$$

Applying Lemma 6.5.2 with $d = \Delta - 1$, we deduce from (6.1.3.1) and (6.1.3.2) that (6.5.3.1) holds, which completes the inner induction.

It remains to check that $\mathrm{ind}_G(z) \neq 0$ if the degree of every vertex v of G is Δ. Let us pick an arbitrary vertex v. We still use (6.1.3.1) and (6.1.3.2), only that the

product in the right hand side of (6.1.3.2) now contains Δ factors. Since the degree of v_i in $G - v - v_1 - \ldots - v_{i-1}$ still does not exceed $\Delta - 1$, we still have (6.5.3.2). From (6.1.3.2) and (6.5.3.2) we conclude that

$$\left| \arg z_v \frac{\mathrm{ind}_{G-v-N_v}(z)}{\mathrm{ind}_{G-v}(z)} \right| \ \leq \ \frac{\pi\Delta}{(2+2\epsilon)(\Delta-1)} + \frac{\epsilon\pi}{2+2\epsilon} \ < \ \pi$$

and $\mathrm{ind}_G(z) \neq 0$ by (6.1.3.1). □

The correlation decay method for complex activities is explored in [H+16].

Our next goal is to prove that among the roots of the univariate independence polynomial nearest to the origin, one is necessarily real and hence negative real [SS05]. More generally, we prove the following result.

6.5.4 Theorem. *Let $G = (V, E)$ be a graph and let $x = (x_v : v \in V)$ be non-negative real activities at the vertices of G, so that $x_v \geq 0$ for all $v \in V$. For $\zeta \in \mathbb{C}$ let us define $\zeta x = (\zeta x_v : v \in V)$ and let*

$$g(\zeta) = \mathrm{ind}_G(\zeta x)$$

be the corresponding univariate polynomial. Then

$$\min_{\zeta\in\mathbb{C}:\ g(\zeta)=0} |\zeta| = \min_{\zeta\in\mathbb{R}:\ g(\zeta)=0} |\zeta|,$$

that is, among the roots of $g(\zeta)$ nearest to the origin, one is negative real.

We follow [Lo12], Sect. 5.3.1. First, we define the chromatic polynomial of a graph.

6.5.5 Lemma. *Let $G = (V, E)$ be a graph without loops or multiple edges. For a positive integer n, let $\mathrm{chr}_G(n)$ be the number of ways to color the vertices of G using a set of at most n distinct colors so that no two vertices spanning an edge are colored with the same color. Then*

1. *For $k = 1, \ldots, |V|$ there exist integer $a_k(G)$ such that*

$$(-1)^{|V|-k} a_k(G) \ \geq \ 0 \quad \textit{for} \quad k = 1, \ldots, |V|$$

and

$$\mathrm{chr}_G(n) = \sum_{k=1}^{|V|} a_k(G) n^k \quad \textit{for all positive integer} \quad n.$$

2. *For $k = 1, \ldots, |V|$ there exist integer $b_k(G)$ such that*

$$b_k(G) \ \geq \ 0 \quad \textit{for} \quad k = 1, \ldots, |V|$$

and

$$\mathrm{chr}_G(n) = \sum_{k=1}^{|V|} b_k(G)\binom{n}{k} \quad \textit{for all positive integer} \quad n.$$

Proof. To prove Part (1), we proceed by induction on the number $|E|$ of edges of G. If $|E| = 0$, that is, if G consists of $|V|$ isolated vertices, then $\mathrm{chr}_G(n) = n^{|V|}$ and the result follows. Suppose now that $|E| > 0$ and let $e \in E$ be an edge of G. Let $G - e$ be the graph with set V of vertices and set $E \setminus \{e\}$ of edges, so that $G - e$ is obtained from G by *deleting* the edge e. Let G/e be the graph obtained from G by *contracting* the edge e. We obtain the set V' of vertices of G/e by replacing the endpoints u, v of e in V by a single new vertex w and we obtain the set E' of edges of G/e by removing e from E and replacing all edges in E with one endpoint in $\{u, v\}$ by the edges with the corresponding endpoint at w (should multiple edges arise, we replace them by a single edge), see Fig. 6.14.

It is not hard to see that

$$\mathrm{chr}_G(n) = \mathrm{chr}_{G-e}(n) - \mathrm{chr}_{G/e}(n) \tag{6.5.5.1}$$

Since the graph $G - e$ has $|V|$ vertices, the graph G/e has $|V| - 1$ vertex and both $G - e$ and G/e contain fewer than $|E|$ edges, the proof follows by induction from (6.5.5.1).

To prove Part (2), we define $b_k(G)$ as the number of ways to color the vertices of G using *exactly* k colors so that no two neighbors are colored with the same color. Clearly $b_k(G) \geq 0$. To color the graph using at most n colors, we choose a subset of k colors in $\binom{n}{k}$ ways and then color the graph in $b_k(G)$ ways using all chosen colors. □

The polynomial chr_G is called the *chromatic polynomial* of the graph G. We can formally define

$$\mathrm{chr}_G(z) = \sum_{k=1}^{|V|} a_k(G) z^k = \sum_{k=1}^{|V|} b_k(G)\binom{z}{k}$$

for any complex $z \in \mathbb{C}$, where

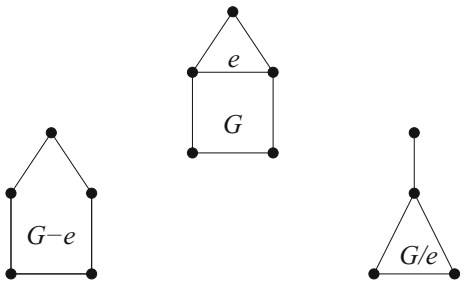


Fig. 6.14 A graph G, its edge e and graphs $G - e$ and G/e

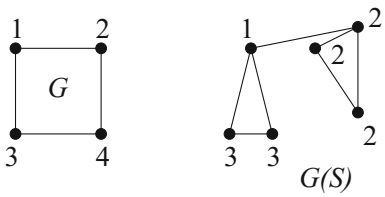


Fig. 6.15 Graphs G and $G(S)$ for $S = \{1, 2, 2, 2, 3, 3\}$

$$\binom{z}{k} = \frac{z(z-1)\cdots(z-k+1)}{k!}.$$

Next, we connect the independence and chromatic polynomials of graphs. Given a graph $G = (V, E)$ and a multiset S of copies of vertices of G (that is, some vertices of G can have multiple copies in S and some can have no copies), we define the graph $G(S)$ with set S of vertices as follows: an edge of $G(S)$ connects two vertices u and v of S if and only if u and v are copies of the same vertex of G or copies of vertices connected by an edge in G, see Fig. 6.15.

If for a multiset S and activities z_v at the vertices of G, we define the monomial

$$\boldsymbol{z}^S = \prod_{v \in S} z_v,$$

where each vertex in S is accounted for with its multiplicity. Our goal is to obtain a power series expansion of $\ln \mathrm{ind}_G(z)$, where $z = (z_v : v \in V)$ is a vector of activities at the vertices of G sufficiently close to 0, so that

$$|1 - \mathrm{ind}_G(z)| < 1.$$

In this case, we choose the branch of $\ln \mathrm{ind}_G(z)$ that is 0 when $z_v = 0$ for all $v \in V$.

6.5.6 Lemma. *Let $G = (V, E)$ be a graph and let $\delta > 0$ be a sufficiently small real number such that*

$$|1 - \mathrm{ind}_G(z)| < 1 \quad \textit{provided} \quad |z_v| \le \delta \quad \textit{for all} \quad v \in V.$$

Then

$$\ln \mathrm{ind}_G(z) = \sum_{S=\{v_1,\ldots,v_1\ldots,v_r,\ldots,v_r\}} \frac{1}{\mu_1! \cdots \mu_r!} a_1(G(S))\mathbf{z}^S,$$

where the sum is taken over multisets S of vertices of G, μ_i is the multiplicity of v_i in S and

$$a_1(G(S)) = \frac{d}{dz} \mathrm{chr}_{G(S)}(z)\Big|_{z=0}$$

is the first coefficient of the chromatic polynomial of $G(S)$. Moreover, the series converges absolutely and uniformly on the polydisc $|z_v| \le \delta$ for $v \in V$.

Proof. Let us fix some $x \in \mathbb{C}$ and consider a function

$$z \longmapsto (1+z)^x = e^{x \ln(1+z)} \quad \text{for} \quad z \in \mathbb{C} \quad \text{such that} \quad |z| < 1,$$

where we choose the branch of $\ln(1+z)$ that is 0 for $z = 0$. We have the Taylor series expansion

$$(1+z)^x = 1 + \sum_{k=1}^{\infty} \binom{x}{k} z^k \quad \text{provided} \quad |z| < 1. \tag{6.5.6.1}$$

Moreover, the series converges absolutely and uniformly on compact sets inside the polydisc $|z| < 1$ and $|x| \leq 1$.

From (6.5.6.1), we get

$$\big(\mathrm{ind}_G(z)\big)^x = 1 + \sum_{k=1}^{\infty} \binom{x}{k} \left(\sum_{\substack{S \subset V, |S|>0 \\ S \text{ independent}}} \mathbf{z}^S \right)^k. \tag{6.5.6.2}$$

Furthermore, we write

$$\left(\sum_{\substack{S \subset V, |S|>0 \\ S \text{ independent}}} \mathbf{z}^S \right)^k = \sum_{\substack{S_1,\ldots,S_k \subset V \\ |S_1|,\ldots,|S_k|>0 \\ S_1,\ldots,S_k \text{ are independent}}} \mathbf{z}^{S_1} \cdots \mathbf{z}^{S_k},$$

where the sum is taken over all ordered k-tuples of not necessarily distinct non-empty independent sets $S_1, \ldots S_k$ of G. Given such a k-tuple $S_1, \ldots, S_k$ of independent sets, let $S = S_1 \sqcup \ldots \sqcup S_k$ be the disjoint union of copies of $S_1, \ldots, S_k$ and let $G(S)$ be the corresponding graph with set S of vertices. Then $G(S)$ can be colored using exactly k colors, so that no two vertices spanning an edge are colored with the same color (we call this a proper k-coloring). Conversely, given a multiset S of possibly multiple copies of vertices of G, each proper k-coloring of $G(S)$ corresponds to a representation $S = S_1 \sqcup \ldots \sqcup S_k$, where $S_1, \ldots, S_k$ are non-empty independent sets in G, as follows: if a copy of a vertex v of G in S is colored with the i-th color then we include v in S_i. If S consists of copies of r distinct vertices $v_1, \ldots, v_r$ with respective multiplicities $\mu_1, \ldots, \mu_r$ then exactly $\mu_1! \cdots \mu_r!$ of proper k-colorings of $G(S)$ correspond to the same ordered k-tuple $S_1, \ldots, S_k$ of non-empty independent sets of G. From (6.5.6.2), we can write

$$\big(\mathrm{ind}_G(z)\big)^x = 1 + \sum_{S=\{v_1,\ldots,v_1,\ldots,v_r,\ldots,v_r\}} \frac{z^S}{\mu_1! \cdots \mu_r!} \sum_{k=1}^{|S|} \binom{x}{k} b_k(G(S)),$$

where the sum is taken over all non-empty multisets S of vertices of G while $b_k(G(S))$ is the number of proper k-colorings of $G(S)$ and $\mu_1, \ldots, \mu_r$ are the multiplicities of vertices in S. From Part (2) of Lemma 6.5.5, we obtain

$$\big(\mathrm{ind}_G(z)\big)^x = 1 + \sum_{S=\{v_1,\ldots,v_1\ldots,v_r,\ldots,v_r\}} \frac{z^S}{\mu_1!\cdots\mu_r!}\,\mathrm{chr}_{G(S)}(x) \tag{6.5.6.3}$$

and the series converges absolutely and uniformly on the polydisc $|z_v| \le \delta$ and $|x| \le 1$, say. On the other hand, for $a > 0$ we can write

$$\ln a = \frac{d}{dz} e^{z \ln a}\Big|_{z=0.}$$

Computing the derivative of (6.5.6.3) at $x = 0$, we obtain

$$\ln \mathrm{ind}_G(z) = \sum_{S=\{v_1,\ldots,v_1,\ldots,v_r,\ldots,v_r\}} \frac{1}{\mu_1!\cdots\mu_r!} a_1(G(S))\mathbf{z}^S,$$

where $a_1(G(S))$ is the first coefficient of the chromatic polynomial of $G(S)$. □

Now we are ready to prove Theorem 6.5.4.

6.5.7 Proof of Theorem 6.5.4. For sufficiently small $\delta > 0$ we have

$$|1 - g(\zeta)| < 1 \quad \text{provided} \quad |\zeta| \le \delta$$

and hence by Lemma 6.5.6, we have a univariate power series expansion

$$\ln g(\zeta) = \sum_{S=\{v_1,\ldots,v_1\ldots,v_r,\ldots,v_r\}} \frac{1}{\mu_1!\cdots\mu_r!} a_1(G(S))\zeta^{|S|}\mathbf{x}^S. \tag{6.5.7.1}$$

It follows then that the distance ρ_0 from 0 to the nearest root of $g(\zeta)$ is the radius of convergence of (6.5.7.1), see also Lemma 2.2.1. Since $x_1, \ldots, x_n \ge 0$, we have

$$\mathbf{x}^S \ge 0 \quad \text{for all} \quad S.$$

By Part (1) of Lemma 6.5.5, we have

$$(-1)^{|S|} a_1(G(S)) \le 0 \quad \text{for all} \quad S.$$

Therefore, the maximum absolute value of the series (6.5.7.1) on any disc $|\zeta| \le \rho$ where it converges is attained at $\zeta = -\rho$ and equal to the sum

$$\sum_{S=\{v_1,\ldots,v_1\ldots,v_r,\ldots,v_r\}} \frac{1}{\mu_1!\cdots\mu_r!} |a_1(G(S))|\,\rho^{|S|}\mathbf{x}^S \tag{6.5.7.2}$$

of non-negative real numbers. In other words, (6.5.7.1) converges in the disc $|\zeta| \leq \rho$ if and only the series of non-negative real numbers (6.5.7.2) converges. Hence the radius ρ_0 of convergence of (6.5.7.1) is the smallest $\rho > 0$ where (6.5.7.2) diverges and $-\rho_0$ is necessarily a root of $g(\zeta)$. □

6.6 On the Local Nature of Independent Sets

Let us compare the correlation decay approach of Sects. 6.3 and 6.4 and the Taylor polynomial interpolation method of Sect. 6.1.5. The correlation decay method is based on the observation that for subcritical activities z_v, the independence polynomial can be approximated based on the local structure of the graph in a neighborhood of each vertex. The Taylor polynomial interpolation method, again for sufficiently small activities, relies on the information about independent sets of a small (logarithmic) size. Such sets can be scattered all over the graph, so it may appear that we rely on some global structural properties of the graph. Here we show that this is an illusion, as the Taylor polynomial interpolation can also be done based on the local information only. Namely, we show that the sum of weights

$$\sum_{\substack{S \subset V \\ S \text{ is independent} \\ |S|=k}} \prod_{v \in S} z_v$$

of independent k-subsets in a graph $G = (V, E; z)$ can be computed entirely from the data contained in the family of $(k-1)$-neighborhoods of the vertices of the graph. Besides, we show that if the maximum degree of a vertex of the graph is bounded above in advance, then the interpolation in Sect. 6.1.5 can be done in genuine polynomial and not just in quasi-polynomial time. Our exposition is loosely based on [PR16].

6.6.1 Definitions A *graph with multiplicities* is an undirected graph $H = (U, R; \mu)$ with set U of vertices, set R of edges, without loops or multiple edges, and with positive integers $\mu(u)$, called multiplicities, assigned to its vertices $u \in U$. We say that two such graphs $H_1 = (U_1, R_1; \mu_1)$ and $H_2 = (U_2, R_2; \mu_2)$ are *isomorphic* if there is a bijection $\phi : U_1 \longrightarrow U_2$, called an *isomorphism*, such that $\{\phi(u), \phi(v)\}$ is an edge of H_2 if and only if $\{u, v\}$ is an edge of H_1 and such that the multiplicity of $\phi(u)$ in H_2 is equal to the multiplicity of u in H_1.

Let $G = (V, E)$ be a graph and let $H = (U, R; \mu)$ be a graph with multiplicities. A map $\psi : U \longrightarrow V$ is called an *embedding* if ϕ is an injection and $\{\phi(u), \phi(v)\}$ is an edge of G if and only if $\{u, v\}$ is an edge of H (multiplicities of vertices of H play no role here). Given a graph $G = (V, E; z)$ with set V of vertices, set E of edges and complex activities z_v at the vertices of G and a graph $H = (U, R; \mu)$ with multiplicities, we define a partition function

$$i_H(G) = \sum_{\substack{\psi\colon U \longrightarrow V \\ \text{is embedding}}} \prod_{u \in U} \left(z_{\psi(u)}\right)^{\mu(u)}.$$

In particular, if F_k is a graph with k vertices, no edges and multiplicity 1 of each vertex, then

$$i_{F_k}(G) = k! \sum_{\substack{S \subset V \\ S \text{ is independent} \\ |S|=k}} \prod_{v \in S} z_v, \tag{6.6.1.1}$$

since every independent k-set of G can be obtained as the image of F_k in exactly $k!$ ways. Note that $i_{F_k}(G)$ is what we need to reconstruct the independence polynomial of G, since

$$\operatorname{ind}_G(z) = 1 + \sum_{k=1}^{|V|} \frac{1}{k!} i_{F_k}(G).$$

6.6.2 Decomposition into connected graphs. Suppose now that the graph $H = (U, R; \mu)$ is connected. Then $i_H(G)$ collects only the local information regarding $G = (V, E)$. Indeed, let u be an arbitrary vertex of H. Once we know the image $\psi(u) \in V$ under the embedding $\psi : U \longrightarrow V$, we know that for every $w \in H$ the image $\psi(w)$ is connected to $\psi(u)$ by a path of m edges in G if and only if w is connected to u in H by a path of m edges. Hence the image of H lies entirely in the $(k-1)$-neighborhood of a vertex of G for $k = |U|$.

The crucial observation is that for any graph H with multiplicities, the value of $i_H(G)$ can be expressed in terms of $i_{H'}(G)$ for connected graphs H' with multiplicities, such that each H' has at most as many vertices as H has and the sum of multiplicities of the vertices of each H' is at most the sum of multiplicities of the vertices of H. Indeed, assuming that H is not connected, let us represent it as a vertex-disjoint union $H = H_1 \cup H_2$ such that there are no edges of H connecting a vertex of H_1 with a vertex of H_2. Expanding the product $i_{H_1}(G) \cdot i_{H_2}(G)$, we observe that we collect all the terms of $i_H(G)$, but also some extra terms, so that

$$i_H(G) = i_{H_1}(G) \cdot i_{H_2}(G) - \sum_{H'} i_{H'}(G), \tag{6.6.2.1}$$

where H' is a graph with multiplicities obtained from H_1 and H_2 by at least one of the following sequence of two operations (a) and (b):

(a) we identify some vertices of H_1 with some vertices of H_2 so that if u_1 is identified with u_2 then the new vertex u of H' is assigned the multiplicity of $\mu(u_1) + \mu(u_2)$; and

(b) we connect some, unchanged on step (a), vertices of H_1 with some, unchanged on step (a), vertices of H_2 by edges.

Whenever we create a multiple edge, we replace it by a single edge. We observe that the number of connected components in each H' so obtained is smaller than the

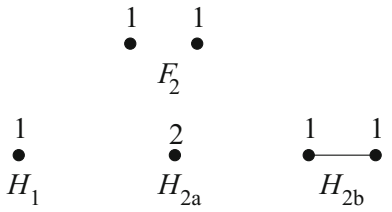


Fig. 6.16 A disconnected graph with multiplicities F_2 and connected graphs with multiplicities H_1, H_{2a} and H_{2b}

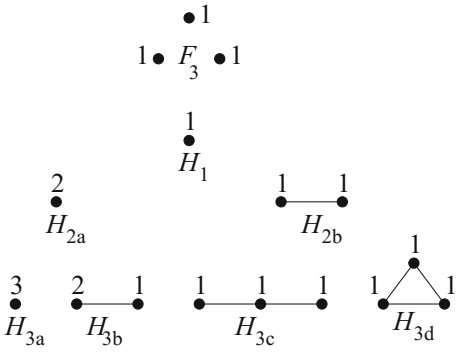


Fig. 6.17 A disconnected graph with multiplicities F_3 and connected graphs with multiplicities H_1, H_{2a}, H_{2b}, H_{3a}, H_{3b}, H_{3c} and H_{3d}

number of connected components of H. Iterating this procedure, we express $i_H(G)$ entirely in terms of $i_{H'}(G)$ with connected H'.

For example, for the graphs with multiplicities pictured on Fig. 6.16, we have

$$i_{F_2}(G) = i_{H_1}(G) \cdot i_{H_1}(G) - i_{H_{2a}}(G) - i_{H_{2b}}(G).$$

A more tedious computation shows that for the graphs with multiplicities pictured on Fig. 6.17, we have

$$\begin{aligned} i_{F_3}(G) =& i_{H_1}(G) \cdot i_{H_1}(G) \cdot i_{H_1}(G) - 3i_{H_{2a}}(G) i_{H_1}(G) - 3i_{H_{2b}}(G) i_{H_1}(G) \\ &+ 2i_{H_{3a}}(G) + 6i_{H_{3b}}(G) + 3i_{H_{3c}}(G) + 2i_{H_{3d}}(G). \end{aligned} \tag{6.6.2.2}$$

6.6.3 The case of a bounded degree. Suppose now that the maximum degree $\Delta(G)$ of a vertex of $G = (V, E)$ is bounded above in advance. Then the procedure of computing (6.6.1.1) can be done in polynomial time $|V|^{O(1)}$, as long as $k = O(\ln |V|)$. The algorithm proceeds as follows.

First, we create a list of connected graphs $H = (U, R; \mu)$ with multiplicities such that there is an embedding $\psi : U \longrightarrow V$ and the sum of weights at the vertices of H does not exceed k. Since H is connected, we can always order the vertices $u_1, \ldots, u_m$, $m \leq k$, of H in such a way that every vertex u_i for $i \geq 2$ has a neighbor among the preceding vertices. Once the image $\psi(u_1)$ is chosen (in at most $|V|$ ways) then for each vertex u_i there are at most $\Delta(G)$ choices of $\psi(u_i)$, given

that the images $u_1, \ldots, u_{i-1}$ are already chosen. This creates a list $\mathcal{H}$ of at most $|V|(\Delta(G))^{k-1} = |V|^{O(1)}$ graphs H. Note that if H has m vertices then there are $\binom{k-1}{m-1}$ ways to assign positive integer weights to the vertices of H so that the sum of weights is k, which is $|V|^{O(1)}$ as long as $k = O(\ln |V|)$.

For each graph H from the list $\mathcal{H}$, we compute $i_H(G)$ in $|V|^{O(1)}$ time.

Next, we create a list $\widehat{\mathcal{H}}$ consisting of the graphs with multiplicities H that are represented as a union of a connected graph from $\mathcal{H}$ and some isolated vertices and such that the sum of multiplicities at the vertices of H does not exceed k. Given a graph $H \in \widehat{\mathcal{H}} \setminus \mathcal{H}$, we write $H = H_1 \cup H_2$, where $H_1 \in \mathcal{H}$ and H_2 consists of the isolated vertices and apply the algorithm of Sect. 6.6.2. Note that all graphs H' in (6.6.2.1) with $i_{H'}(G) \neq 0$ (we only need to collect those) also belong to $\widehat{\mathcal{H}}$ and have fewer isolated vertices than H has. When applying (6.6.2.1), we should account for isomorphic graphs H' (this is how we get integer coefficients in the formula (6.6.2.2)). However, testing the isomorphism of two graphs $H'_1, H'_2 \in \widehat{\mathcal{H}}$ reduces to testing the isomorphism of their connected components $H_1, H_2 \in \mathcal{H}$, which can be done in $|V|^{O(1)}$ time as above: once we picked the image of a vertex of H_1 under a prospective isomorphism ϕ, we have at most $\Delta(G)$ choices for the image of each next vertex. Thus we recursively compute $i_H(G)$ for all $H \in \widehat{\mathcal{H}}$ in the order of the increasing number of isolated vertices, so that in the end we compute (6.6.1.1).

Chapter 7
The Graph Homomorphism Partition Function

Known in statistical physics as the partition function of a multi-spin system, this is one of the most general forms of a partition function. It covers permanents, hafnians, independent sets, graph colorings and some more exotic objects such as the Hamiltonian permanent. We apply the Taylor polynomial interpolation to find a domain where the partition function can be efficiently approximated. This leads to "softer" (doable) versions of "hard" (impossible) problems of combinatorial enumeration: for example, instead of counting all independent sets of a given cardinality in a graph, we compute the total weight of all subsets of vertices of a given cardinality, where the weight of each subset is exponentially small in the number of edges of the graph it spans. We discuss one of the oldest and most famous models in statistical physics, the Ising model for magnetization, which connects various topics: perfect matchings, graph homomorphisms, cuts in graphs and phase transitions of various kinds. The Lee–Yang Theorem asserts that the zeros of the partition function of cuts lie on the unit circle, which is interpreted as the absence of phase transition in the presence of a magnetic field.

7.1 The Graph Homomorphism Partition Function

7.1.1 Definition. Let $G = (V, E)$ be an undirected graph with set V of vertices, set E of edges, without multiple edges or loops, and let $A = \left(a_{ij}\right)$ be a $k \times k$ symmetric real or complex matrix. We define the *graph homomorphism partition function* by

$$\hom_G(A) = \sum_{\phi: V \longrightarrow \{1,\ldots,k\}} \prod_{\{u,v\} \in E} a_{\phi(u)\phi(v)}. \tag{7.1.1.1}$$

The sum is taken over all maps ϕ of the set V of vertices into the set $\{1, \ldots, k\}$ of indices of the matrix entries and the product is taken over all edges of the graph G.

A. Barvinok, *Combinatorics and Complexity of Partition Functions*,
Algorithms and Combinatorics 30, DOI 10.1007/978-3-319-51829-9_7

If A is the adjacency matrix of a graph H with set U of k vertices then $\hom_G(A)$ counts *graph homomorphisms*, that is, maps $\phi : V \longrightarrow U$ such that $\{\phi(u), \phi(v)\}$ is an edge of H whenever $\{u, v\}$ is an edge of G.

Choosing the matrix A in a special way, we obtain various quantities of interest.

7.1.2 Example: independent sets. Let us choose $k = 2$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Each map $\phi : V \longrightarrow \{1, 2\}$ defines a set of vertices $S_\phi = \phi^{-1}(1) \subset V$ and the contribution of ϕ in (7.1.1.1) is 1 if S is an independent set in G and 0 otherwise. Hence $\hom_G(A)$ counts independent sets in G.

7.1.3 Example: colorings. Let us define

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

We interpret every map $\phi : V \longrightarrow \{1, \ldots, k\}$ as a coloring of the vertices of G into one of the k colors. Then the contribution of ϕ in (7.1.1.1) is 1 if the coloring is *proper*, that is, the endpoints of every edge are colored differently, and the contribution of ϕ is 0 otherwise. Hence $\hom_G(A)$ counts the proper colorings of G with k colors.

For more examples, see Sect. 5.3 of [Lo12].

Recall that by $\Delta(G)$ we denote the largest degree of a vertex of G.

7.1.4 Theorem. *For a positive integer Δ, let*

$$\delta_\Delta = \sin \frac{\alpha}{2} \cos \frac{\alpha \Delta}{2}$$

for some $\alpha = \alpha_\Delta$ such that

$$0 < \alpha < \frac{2\pi}{3\Delta},$$

so that we can choose $\delta_3 = 0.18$, $\delta_4 = 0.13$ and $\delta_\Delta = \Omega(1/\Delta)$. Then for any graph G with $\Delta(G) \leq \Delta$, we have

$$\hom_G(Z) \neq 0$$

for any $k \times k$ complex symmetric matrix $Z = \left(z_{ij}\right)$ such that

$$\left|1 - z_{ij}\right| \leq \delta_\Delta \quad \text{for all} \quad 1 \leq i, j \leq k.$$

A version of Theorem 7.1.4 was first proved in [BS14]. We present a simpler proof achieving better constants.

As before, see Sect. 3.6, Theorem 4.1.5, Sects. 4.4, 5.5 and 6.1.5, we obtain that $\hom_G(A)$ is easily computable if the entries a_{ij} satisfy a slightly stronger inequality.

7.1.5 Theorem. *Let us fix a constant $0 < \delta < \delta_\Delta$, where δ_Δ is the constant in Theorem 7.1.4. Then there exists a $\gamma = \gamma(\delta_\Delta/\delta) > 0$ and for any $0 < \epsilon < 1$, for any graph $G = (V, E)$ such that $\Delta(G) \le \Delta$ and any k there exists a polynomial $p = p_{G,k,\delta,\epsilon}$ in the entries of $k \times k$ symmetric matrix $A = \left(a_{ij}\right)$ such that*

$$\deg p \ \le \ \gamma\left(\ln|E| - \ln\epsilon\right)$$

and

$$\left|\ln \hom_G(A) - p(A)\right| \ \le \ \epsilon$$

provided

$$\left|1 - a_{ij}\right| \ \le \ \delta \quad \textit{for all} \quad 1 \le i, j \le k.$$

Moreover, given δ, G, $\epsilon > 0$ and k, the polynomial p can be constructed in $(k|E|)^{O(\ln|E| - \ln\epsilon)}$ time, where the implied constant in the "O" notation depends on the ratio δ_Δ/δ alone. The proof is very similar to that of Theorem 3.6.2, we sketch it below.

Let $J = J_k$ be the $k \times k$ matrix filled with 1s. We define a univariate polynomial $g = g_{G,A}$ by

$$g(z) = \hom_G\left(J + z\left(A - J\right)\right),$$

so that $g(0) = \hom_G(J) = k^{|V|}$ and $g(1) = \hom_G(A)$. We note that

$$\begin{aligned}\frac{d^s}{dz^s} g(z)\Big|_{z=0} &= \frac{d^s}{dz^s} \sum_{\phi: V \longrightarrow \{1,\ldots,k\}} \prod_{\{u,v\} \in E} \left(1 + z\left(a_{\phi(u)\phi(v)} - 1\right)\right)\Big|_{z=0} \\ &= \sum_{\phi: V \longrightarrow \{1,\ldots,k\}} \sum_{\{u_1,v_1\},\ldots,\{u_s,v_s\} \in E} \left(a_{\phi(u_1)\phi(v_1)} - 1\right) \cdots \left(a_{\phi(u_s)\phi(v_s)} - 1\right),\end{aligned}$$

where the inner sum is taken over all ordered s-tuples I of distinct edges of G. Let $V(I)$ be the set of distinct vertices among $u_1, v_1, \ldots, u_s, v_s$. Then we can further write

$$\begin{aligned}\frac{d^s}{dz^s} g(z)\Big|_{z=0} = &\sum_{\substack{I = (\{u_1,v_1\},\ldots,\{u_s,v_s\}) \\ \{u_1,v_1\},\ldots,\{u_s,v_s\} \in E}} k^{|V| - |V(I)|} \\ &\times \sum_{\phi: V(I) \longrightarrow \{1,\ldots,k\}} \left(a_{\phi(u_1)\phi(v_1)} - 1\right) \cdots \left(a_{\phi(u_s)\phi(v_s)} - 1\right).\end{aligned}$$

Here the factor of $k^{|V| - |V(I)|}$ accounts for the number of ways to extend a map $\phi : V(I) \longrightarrow \{1, \ldots, k\}$ to the whole set $V \supset V(I)$ of vertices. It follows now that $g^{(s)}(0)$ is a polynomial of degree s in the entries a_{ij} computable in $(|E|k)^{O(s)}$ time.

We define $f(z) = \ln g(z)$ in a neighborhood of $z = 0$ and the proof proceeds as in Sect. 3.6.7.

Patel and Regts show [PR16] that if $\Delta(G)$ is fixed in advance, then the value of $p(A)$ can be computed in polynomial time $(k|E|/\epsilon)^{O(1)}$, where the implied constant depends on the ratio δ_Δ/δ only.

Using Theorem 7.1.5, we obtain the following relaxed versions of hard counting problems in Examples 7.1.2 and 7.1.3.

7.1.6 Example: sets weighted by independence. In the context of Example 7.1.2, let us define A by

$$A = \begin{pmatrix} 1-\delta & 1+\delta \\ 1+\delta & 1+\delta \end{pmatrix},$$

where δ is the constant of Theorem 7.1.5. Then

$$\begin{aligned} (1+\delta)^{-|E|} \hom_G(A) &= \sum_{S \subset V} w(S) \\ \text{where} \quad w(S) &= (1+\delta)^{-e(S)} (1-\delta)^{e(S)} \end{aligned} \tag{7.1.6.1}$$

and $e(S)$ is the number of edges in G with both endpoints in S. In particular, $w(S) = 1$ if S is independent and

$$\exp\left\{-2\delta e(S) - \delta^3 e(S)\right\} \;\le\; w(S) \;\le\; \exp\left\{-2\delta e(S)\right\}.$$

Hence (7.1.6.1) is the sum over all subsets of vertices of G, where each subset S is counted with weight 1 if S is independent and is counted with a weight exponentially small in the number of edges that vertices of S span, if S is not independent.

As follows by Theorem 7.1.5, we can compute the sum (7.1.6.1) in quasi-polynomial time (genuinely polynomial time, if $\Delta(G)$ is fixed in advance [PR16]).

7.1.7 Colorings weighted by properness. In the context of Example 7.1.3, let us define

$$a_{ij} = \begin{cases} 1+\delta & \text{if } i \neq j \\ 1-\delta & \text{if } i = j, \end{cases}$$

where $\delta > 0$ is the constant in Theorem 7.1.5. Then

$$\begin{aligned} (1+\delta)^{-|E|} \hom_G(A) &= \sum_{\phi: V \longrightarrow \{1,\dots,k\}} w(\phi) \\ \text{where} \quad w(\phi) &= (1+\delta)^{-e(\phi)} (1-\delta)^{e(\phi)} \end{aligned} \tag{7.1.7.1}$$

and $e(\phi)$ is the number of edges of G whose both endpoints are colored into the same color by the coloring ϕ. Thus we have $w(\phi) = 1$ if ϕ is a proper coloring and

$$\exp\left\{-2\delta e(\phi) - \delta^3 e(\phi)\right\} \;\le\; w(\phi) \;\le\; \exp\left\{-2\delta e(\phi)\right\}.$$

Hence (7.1.7.1) represents the sum over all colorings ϕ of G, where ϕ is counted with weight 1 if ϕ is proper and is counted with a weight exponentially small in the number of edges that are not properly colored, if ϕ is not proper.

Theorem 7.1.5 implies that we can compute the sum (7.1.7.1) in quasi-polynomial time.

To prove Theorem 7.1.4, we first introduce a multi-affine version of $\hom_G$.

7.1.8 Edge-colored graph homomorphisms. Let $G = (V, E)$ be a graph as above and let $Z = \left(z_{ij}^{uv}\right)$ be a $|E| \times \frac{k(k+1)}{2}$ complex matrix with entries indexed by edges $\{u, v\} \in E$ and unordered pairs $1 \le i, j \le k$. We write z_{ij}^{uv} instead of $z_{\{i,j\}}^{\{u,v\}}$ assuming that

$$z_{ij}^{uv} = z_{ji}^{uv} = z_{ji}^{vu} = z_{ij}^{vu}.$$

Equivalently, we can think that a $k \times k$ symmetric matrix is attached to every edge of G. We introduce the partition function

$$\mathrm{Hom}_G(Z) = \sum_{\phi: V \longrightarrow \{1,\dots,k\}} \prod_{\{u,v\} \in E} z_{\phi(u)\phi(v)}^{uv},$$

which we call the *partition function of edge-colored homomorphisms*. If $z_{ij}^{uv} = z_{ij}$ (that is, the same symmetric matrix is attached to each edge of G), we are in the situation of Definition 7.1.1 and $\mathrm{Hom}_G(Z) = \hom_G(Z)$. The advantage of working with $\mathrm{Hom}_G(Z)$ as opposed to $\hom_G(Z)$ is that $\mathrm{Hom}_G(Z)$ is a *multi-affine* function, that is, the degree of $\mathrm{Hom}_G(Z)$ in each variable z_{ij}^{uv} does not exceed 1.

We will prove that in fact

$$\mathrm{Hom}_G(Z) \neq 0 \quad \text{provided} \quad \left|1 - z_{ij}^{uv}\right| \le \delta_\Delta$$
$$\text{for all} \quad \{u, v\} \in E \quad \text{and all} \quad 1 \le i, j \le k,$$

where δ_Δ is the constant of Theorem 7.1.4.

7.1.9 The recursion. For a sequence $W = (v_1, \dots, v_r)$ of distinct vertices of G and a sequence $L = (l_1, \dots, l_r)$ of not necessarily distinct indices $1 \le l_1, \dots, l_r \le k$, we define

$$\mathrm{Hom}_L^W(Z) = \sum_{\substack{\phi: V \longrightarrow \{1,\dots,k\} \\ \phi(v_1)=l_1,\dots,\phi(v_r)=l_r}} \prod_{\{u,v\} \in E} z_{\phi(u)\phi(v)}^{uv}$$

(we suppress the graph G in the notation). In words: we restrict the sum defining Hom_G to the maps ϕ that map prescribed vertices to prescribed indices. We denote by $|W|$ and by $|L|$ the number of vertices in W and the number of indices in L respectively. If W is a sequence of distinct vertices and L is a sequence of not necessarily distinct indices such that $|W| = |L|$, for a vertex w in W we denote by $l(w)$ the corresponding index in L, so $l(v_i) = l_i$ in the above definition. We denote

by (W, u) the sequence W appended by a vertex u, different from all vertices in W and by (L, l) the sequence of indices L appended by an index l, not necessarily different from the indices in L. Then, for any vertex u not in the sequence W, we have

$$\mathrm{Hom}_L^W(Z) = \sum_{l=1}^{k} \mathrm{Hom}_{(L,l)}^{(W,u)}(Z). \tag{7.1.9.1}$$

For a $0 < \delta < 1$ we define the polydisc $\mathcal{U}(\delta) = U(\delta, G)$ consisting of all $|E| \times \frac{k(k+1)}{2}$ matrices $Z = \left(z_{ij}^{uv}\right)$ such that

$$\left|1 - z_{ij}^{uv}\right| \leq \delta \quad \text{for all} \quad \{u, v\} \in E \quad \text{and} \quad 1 \leq i, j \leq k.$$

We will use the following straightforward observation: suppose that W contains two vertices u and v such that $\{u, v\} \in E$ with corresponding indices l and m in L, so that $W = (W', u, v)$ and $L = (L', l, m)$. Let $A, B \in \mathcal{U}(\delta)$ be two matrices that differ only in the entries z_{ij}^{uv} for $1 \leq i, j \leq k$. Then

$$\mathrm{Hom}_L^W(A) = \frac{a_{lm}^{uv}}{b_{lm}^{uv}} \mathrm{Hom}_L^W(B).$$

In particular, if $\mathrm{Hom}_L^W(A) \neq 0$ and $\mathrm{Hom}_L^W(B) \neq 0$ then the angle between non-zero complex numbers $\mathrm{Hom}_L^W(A)$ and $\mathrm{Hom}_L^W(B)$ does not exceed $2 \arcsin \delta$, see Fig. 3.7.

7.1.10 Proof of Theorem 7.1.4**.** Let us denote δ_Δ just by δ and let

$$0 < \alpha < \frac{2\pi}{3\Delta}$$

be a number such that

$$\delta = \sin \frac{\alpha}{2} \cos \frac{\alpha\Delta}{2}.$$

We prove by the descending induction on $r = |V|, \ldots, 1$ the following statements:

Statement $1.r$. Let W be a sequence of distinct vertices and let L be a sequence of not necessarily distinct indices such that $|W| = |L| = r$. Then $\mathrm{Hom}_L^W(Z) \neq 0$ for all $Z \in \mathcal{U}(\delta)$.

Statement $2.r$. Let W be a sequence of distinct vertices such that $|W| = r$. Suppose that $W = (W', u)$ and let L' be a sequence of not necessarily distinct indices such that $|W'| = |L'| = r - 1$. Let $1 \leq l, m \leq k$ be indices. Then for any $Z \in \mathcal{U}(\delta)$ the angle between complex numbers $\mathrm{Hom}_{(L',l)}^{(W',u)}(Z) \neq 0$ and $\mathrm{Hom}_{(L',m)}^{(W',u)}(Z) \neq 0$ does not exceed $\alpha\Delta$.

Statement 3.r. Let W be a sequence of distinct vertices and let L be a sequence of not necessarily distinct indices such that $|W| = |L| = r$ and suppose that $W = (W', u)$ and $L = (L', l)$. Let v be a vertex not from W and let $A, B \in \mathcal{U}(\delta)$ be two matrices that differ only in the coordinates z_{lj}^{uv} for $j = 1, \ldots, k$. Then the angle between $\mathrm{Hom}_L^W(A) \neq 0$ and $\mathrm{Hom}_L^W(B) \neq 0$ does not exceed α.

Suppose that $r = |V|$ so that W is a sequence of all vertices V of G. If L is a sequence of indices such that $|L| = |V|$ then

$$\mathrm{Hom}_L^W(Z) = \prod_{\{u,v\} \in E} z_{l(u)\, l(v)}^{u\ v} \neq 0$$

and Statement 1.r follows. Writing $W = (W', u)$, we have

$$\mathrm{Hom}_{(L',l)}^{(W',u)}(Z) = \left(\prod_{v:\ \{u,v\} \in E} \frac{z_{l\, l(v)}^{u\ v}}{z_{m\, l(v)}^{u\ v}} \right) \mathrm{Hom}_{(L',m)}^{(W',u)}(Z)$$

and hence the angle between $\mathrm{Hom}_{(L',l)}^{(W',u)}(Z) \neq 0$ and $\mathrm{Hom}_{(L',m)}^{(W',u)}(Z) \neq 0$ does not exceed

$$2\Delta \arcsin \delta \ \leq \ \alpha\Delta$$

and Statement 2.r follows. Statement 3.r is vacuous since there are no vertices outside of W.

Suppose that $1 \leq r < |V|$ and that Statements 1.$(r+1)$, 2.$(r+1)$ and 3.$(r+1)$ hold.

Let W be a sequence of distinct vertices and let L be a sequence of not necessarily distinct indices such that $|W| = |L| = r$. Let us choose a vertex v not in W. Then by (7.1.9.1),

$$\mathrm{Hom}_L^W(Z) = \sum_{j=1}^{k} \mathrm{Hom}_{(L,j)}^{(W,v)}(Z) \tag{7.1.10.1}$$

From Statement 1.$(r+1)$ we have $\mathrm{Hom}_{(L,j)}^{(W,v)}(Z) \neq 0$ for all $j = 1, \ldots, k$ and from Statement 2.$(r+1)$ the angle between any two complex numbers $\mathrm{Hom}_{(L,i)}^{(W,v)}(Z) \neq 0$ and $\mathrm{Hom}_{(L,j)}^{(W,v)}(Z) \neq 0$ does not exceed $\alpha\Delta < 2\pi/3$. Therefore, by Lemma 3.6.4, we have

$$\mathrm{Hom}_L^W(Z) \neq 0$$

and Statement 1.r follows.

Let W and L with $|W| = |L| = r$ be sequences as above and suppose that $W = (W', u)$ and $L = (L', l)$. Let v be a vertex not in W and let $A, B \in \mathcal{U}(\delta)$ be the matrices that differ only in the coordinates z_{lj}^{uv} for $j = 1, \ldots, k$. Let us define a matrix $C \in \mathcal{U}(\delta)$ such that

$$c_{lj}^{uv} = 1 \quad \text{for} \quad j = 1, \ldots, k$$

and C agrees with A and B in all other entries. From Statement 1.$(r+1)$ we have

$$\mathrm{Hom}_{(L,j)}^{(W,v)}(C) \neq 0 \quad \text{for} \quad j = 1, \ldots, k$$

and from Statement 2.$(r+1)$ the angle between any two numbers $\mathrm{Hom}_{(L,i)}^{(W,v)}(C) \neq 0$ and $\mathrm{Hom}_{(L,j)}^{(W,v)}(C) \neq 0$ does not exceed $\alpha\Delta < 2\pi/3$. We rewrite (7.1.10.1) as

$$\mathrm{Hom}_L^W(A) = \sum_{j=1}^{k} a_{lj}^{uv} \mathrm{Hom}_{(L',l,j)}^{(W',u,v)}(C) \quad \text{and} \quad \mathrm{Hom}_L^W(B) = \sum_{j=1}^{k} b_{lj}^{uv} \mathrm{Hom}_{(L',l,j)}^{(W',u,v)}(C).$$

Applying Lemma 3.6.4 again, we conclude that the angle between $\mathrm{Hom}_L^W(A) \neq 0$ and $\mathrm{Hom}_L^W(B) \neq 0$ does not exceed

$$2 \arcsin \frac{\delta}{\cos \frac{\alpha\Delta}{2}} = \alpha$$

and Statement 3.r holds.

Let W with $|W| = r$ be a sequence as above and suppose that $W = (W', u)$. Let L' be a sequence of indices such that $|L'| = r - 1$. Given $A \in \mathcal{U}(\delta)$ and two indices $1 \leq l, m \leq k$, let us define a matrix B by

$$b_{lj}^{uv} = a_{mj}^{uv} \quad \text{for all} \quad v \quad \text{such that} \quad \{u, v\} \in E \quad \text{and all} \quad j = 1, \ldots, k$$

and keeping all other entries of B the same as in A. Then

$$\mathrm{Hom}_{(L',l)}^{(W',u)}(B) = \mathrm{Hom}_{(L',m)}^{(W',u)}(A).$$

Let d_0 be the number of neighbors of u in the sequence W' and let d_1 be the number of neighbors of u not in the sequence W'. Then, from Statement 1.r we have $\mathrm{Hom}_{(L',l)}^{(W',u)}(A) \neq 0$ and $\mathrm{Hom}_{(L',m)}^{(W',u)}(A) \neq 0$ while from Statement 3.r the angle between $\mathrm{Hom}_{(L',l)}^{(W',u)}(A)$ and $\mathrm{Hom}_{(L',m)}^{(W',u)}(A) = \mathrm{Hom}_{(L',l)}^{(W',u)}(B)$ does not exceed

$$2d_0 \arcsin \delta + d_1\alpha \;\leq\; d_0\alpha + d_1\alpha = \alpha\Delta,$$

which proves Statement 2.r.

This concludes the induction and hence the proof of Statements 1.1 and 2.1. For any vertex v of V, we have

$$\mathrm{Hom}_G(Z) = \sum_{j=1}^{k} \mathrm{Hom}_j^v(Z)$$

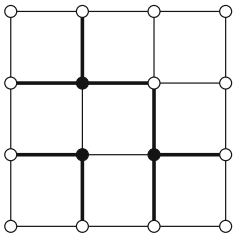


Fig. 7.1 A graph and a cut of 8 edges associated with the set S of 3 *black dots*

and the proof of the theorem follows by Statement 1.1, Statement 1.2 and Lemma 3.6.4. □

7.1.11 Cuts and limits of approximability. Let $G = (V, E)$ be a graph and let $S \subset V$ be a set of vertices. The *cut* associated with S is the set of all edges of G with one endpoint in S and the other not in S. We denote by $\text{cut}_G(S)$ the number of edges in the cut. For example, for the graph G and set S in Fig. 7.1, we have $\text{cut}_G(S) = 8$.

Let

$$\mu(G) = \max_{S \subset V} \text{cut}_G(S)$$

be the largest number of edges in a cut of a graph G. Berman and Karpinski proved [BK99] that there is an absolute constant $\beta > 1$ such that it is an NP-hard problem to approximate $\mu(G)$ within a factor $\beta > 1$ for a given graph satisfying $\Delta(G) \leq 3$. Clearly, the problem remains NP-hard if we further restrict it to connected graphs, in which case $\mu(G) \geq |V| - 1$.

Let $k = 2$, let us choose $0 < \epsilon < 1$ and let

$$A_\epsilon = \begin{pmatrix} \epsilon & 1 \\ 1 & \epsilon \end{pmatrix}.$$

Then

$$\epsilon^{-|E|} \text{hom}_G(A_\epsilon) = \sum_{S \subset V} \epsilon^{-\text{cut}_G(S)}.$$

Since the number of terms in the above sum is $2^{|V|}$, we obtain

$$\frac{\ln \text{hom}_G(A_\epsilon)}{\ln(1/\epsilon)} + |E| - |V| \frac{\ln 2}{\ln(1/\epsilon)} \leq \mu(G) \leq \frac{\ln \text{hom}_G(A_\epsilon)}{\ln(1/\epsilon)} + |E|.$$

Assuming now that G is a connected graph with $\Delta(G) \leq 3$, we conclude that for any given $\delta > 0$, by choosing a sufficiently small $\epsilon = \epsilon(\delta) > 0$, we approximate $\mu(G)$ within a relative error δ by

$$|E| + \frac{\ln \text{hom}_G(A_\epsilon)}{\ln(1/\epsilon)}.$$

Hence for some $\epsilon_0 > 0$ approximating $\hom_G(A_{\epsilon_0})$ is an NP-hard problem. This can be contrasted with Theorem 3.7.1, where approximability holds for matrices with positive entries arbitrarily close to 0.

For hardness results on exact computation of $\hom_G$, see [BG05] and [C+13], for hardness of approximate computation, see [GJ12] and [GJ15].

For applications of the correlation decay approach to approximating Hom_G, see [LY13].

Closely related *edge-coloring models*, also known as *vertex models*, *holant problems* or *tensor networks* were studied in [Re15] and [PR16]. There we consider all possible colorings ϕ of the edges of $G = (V, E)$ into k colors, at each vertex v of G a complex number $z(v, \phi)$ is determined by the multiset of the numbers of edges of each color that have v as an endpoint, and the partition function computes

$$\sum_{\phi} \prod_{v \in V} z(v, \phi).$$

It is shown that the partition function is never zero provided

$$|1 - z(v, \phi)| \leq \frac{0.35}{\Delta(G) + 1} \quad \text{for all} \quad v \in V \quad \text{and all} \quad \phi,$$

which leads to a quasi-polynomial [Re15] and polynomial [PR16] in the case of a bounded degree $\Delta(G)$ algorithms for approximating the partition function in the corresponding domains.

7.2 Sharpening in the Case of a Positive Real Matrix

In this section, we sharpen the approximation bounds in Theorem 7.1.5, assuming that the matrix A is positive real.

7.2.1 Theorem. *Let*

$$\delta_3 = \tan \frac{\pi}{9} \approx 0.36 \quad \textit{and} \quad \delta_\Delta = \tan \frac{\pi}{4\,(\Delta - 1)} \quad \textit{for integer} \quad \Delta \geq 4.$$

so that $\delta_4 \approx 0.27$, $\delta_5 \approx 0.20$, *etc.*

Let us fix

$$0 \;\leq\; \delta \;<\; \delta_\Delta.$$

Then there exists a constant $\gamma = \gamma\,(\delta_\Delta/\delta) > 0$ *and for every connected graph* $G = (V, E)$, *for every positive integer* k *and every* $0 < \epsilon < 1$ *there is a polynomial* $p = p_{G,k,\delta,\epsilon}$ *in the entries of a* $k \times k$ *symmetric matrix* A *such that*

$$\deg p \;\leq\; \gamma\,(\ln |E| - \ln \epsilon)$$

and

$$|\ln \hom_G(A) - p(A)| \ \leq \ \epsilon$$

for any $k \times k$ real symmetric matrix $A = \left(a_{ij}\right)$ such that

$$\left|1 - a_{ij}\right| \ \leq \ \delta \quad \textit{for all} \quad 1 \leq i, j \leq k$$

provided $\Delta(G) \leq \Delta$.

As in Sect. 7.1, given δ, G, $\epsilon > 0$ and k, the polynomial p can be constructed in $(k|E|)^{O(\ln|E| - \ln\epsilon)}$ time, where the implied constant in the "O" notation depends on the ratio δ_Δ/δ alone. If Δ is fixed in advance, the value of $p(A)$ can be computed in polynomial time $(k|E|/\epsilon)^{O(1)}$, where the implied constant in the "O" notation depends on the ratio δ_Δ/δ alone, cf. [PR16].

As in Sects. 3.7 and 4.4, we deduce Theorem 7.2.1 by bounding the complex roots of $\hom_G$ away from the positive real axis. As in Sect. 7.1, it is more convenient to work with the multi-affine extension Hom_G, see Sect. 7.1.8. We deduce Theorem 7.2.1 from the following result.

7.2.2 Theorem. *For $\Delta \geq 3$, let δ_Δ be the constant of Theorem 7.2.1 and let us choose*

$$0 \ \leq \ \delta \ < \ \delta_\Delta.$$

Let

$$\tau = (1-\delta)\sin\left(\frac{\pi}{18} - \frac{1}{2}\arctan\delta\right) \ > \ 0 \quad \textit{if} \quad \Delta = 3 \quad \textit{and}$$

$$\tau = (1-\delta)\sin\left(\frac{\pi}{8\,(\Delta-1)} - \frac{1}{2}\arctan\delta\right) > 0 \quad \textit{for} \quad \Delta \geq 4.$$

Then for any connected graph G such that $\Delta(G) \leq \Delta$, we have

$$\mathrm{Hom}_G(Z) \neq 0$$

for any $k \times k$ complex symmetric matrix $Z = \left(z_{ij}\right)$ such that

$$\left|1 - \Re\, z_{ij}\right| \ \leq \ \delta \quad \textit{and} \quad \left|\Im\, z_{ij}\right| \ \leq \ \tau \quad \textit{for all} \quad 1 \leq i, j \leq k.$$

For the rest of the section, we prove Theorem 7.2.2. Theorem 7.2.1 follows then as in Sects. 3.7 and 4.4.

As in Sect. 7.1.9, we define restricted functionals $\mathrm{Hom}_L^W(Z)$. For $0 \leq \delta < 1$ and $0 < \tau < 1 - \delta$, we define a domain $\mathcal{U}(\delta, \tau) = \mathcal{U}(\delta, \tau, G)$ in the space of matrices Z:

$$\mathcal{U}(\delta,\tau) = \Big\{ Z = \left(z_{ij}^{uv}\right) : \ \left|1 - \Re\, z_{ij}^{uv}\right| \ \leq \ \delta, \ \left|\Im\, z_{ij}^{uv}\right| \ \leq \ \tau \\ \text{for all} \quad \{u, v\} \in E \quad \text{and all} \quad 1 \leq i, j \leq n \Big\}.$$

We will use the following observation. Let W be a sequence of distinct vertices, which includes some two vertices u and v such that $\{u, v\}$ is an edge of G and let L be a sequence of not necessarily distinct indices such that $|W| = |L|$. Let $A, B \in \mathcal{U}(\delta, \tau)$ be two matrices that differ only in the entries z_{ij}^{uv} for $1 \le i, j \le k$. Then

$$\mathrm{Hom}_L^W(B) = \frac{b^{u\ v}_{l(u)l(v)}}{a^{u\ v}_{l(u)l(v)}} \mathrm{Hom}_L^W(A),$$

where $l(u)$ and $l(v)$ are the indices in L corresponding to u and v in W. In particular, if $\mathrm{Hom}_L^W(A)$ and $\mathrm{Hom}_L^W(B) \neq 0$ then the angle between two numbers is at most

$$2 \arctan \frac{\tau}{1-\delta}.$$

7.2.3 Proof of Theorem 7.2.2**.** Let

$$\alpha = \frac{\pi}{9} + \arctan \delta \quad \text{if} \quad \Delta = 3 \quad \text{and}$$
$$\alpha = \frac{\pi}{4(\Delta - 1)} + \arctan \delta \quad \text{if} \quad \Delta \ge 4.$$

We introduce the following statements.

Statement $1.r$. Let W be a sequence of distinct vertices such that the graph induced on W is connected and let L be a sequence of not necessarily distinct indices such that $|W| = |L| = r$. Then

$$\mathrm{Hom}_L^W(Z) \neq 0 \quad \text{for all} \quad Z \in \mathcal{U}(\delta, \tau).$$

Statement $2.r$. Let W be a sequence of distinct vertices such that the graph induced on W is connected and $|W| = r$. Suppose that $W = (W', u)$ and let L' be a sequence of not necessarily distinct indices such that $|L'| = r - 1$. Then for any two indices $1 \le l, m \le k$ and any $Z \in \mathcal{U}(\delta, \tau)$ the angle between complex numbers

$$\mathrm{Hom}_{(L',l)}^{(W',u)}(Z) \neq 0 \quad \text{and} \quad \mathrm{Hom}_{(L',m)}^{(W',u)}(Z) \neq 0$$

does not exceed $\pi/2$.

Statement $3.r$. Let W be a sequence of distinct vertices such that the graph induced on W is connected and let L be a sequence of not necessarily distinct indices such that $|W| = |L| = r$. Suppose that $W = (W', u)$ and $L = (L', l)$ and let v be a neighbor of u not in the sequence W. Let $A, B \in \mathcal{U}(\delta, \tau)$ be two matrices that differ only in the entries z_{lj}^{uv} where $\{u, v\} \in E$ and $j = 1, \ldots, k$. Then the angle between

$$\mathrm{Hom}_L^W(A) \neq 0 \quad \text{and} \quad \mathrm{Hom}_L^W(B) \neq 0$$

does not exceed α.

First, we claim that Statements $1.r$, $2.r$ and $3.r$ hold for $r = |V|$. Indeed, suppose that $r = |V|$, so that W is a sequence consisting of all vertices of the graph. Then

$$\mathrm{Hom}_L^W(Z) = \prod_{\{u,v\}\in E} z^{u\ v}_{l(u)l(v)} \neq 0$$

and Statement $1.r$ follows. Writing $W = (W', u)$, we have

$$\mathrm{Hom}^{(W',u)}_{(L',l)}(Z) = \prod_{v:\ \{u,v\}\in E} \frac{z^{u\ v}_{l\ l(v)}}{z^{u\ v}_{m\ l(v)}} \mathrm{Hom}^{(W',u)}_{(L',m)}(Z)$$

and hence the angle between $\mathrm{Hom}^{(W',u)}_{(L',l)}(Z) \neq 0$ and $\mathrm{Hom}^{(W',u)}_{(L',m)}(Z) \neq 0$ does not exceed

$$2\Delta \arctan \frac{\tau}{1-\delta},$$

which does not exceed

$$\frac{\pi}{3} < \frac{\pi}{2} \quad \text{if} \quad \Delta = 3$$

and does not exceed

$$\frac{\pi\Delta}{4(\Delta-1)} < \frac{\pi}{2} \quad \text{if} \quad \Delta \geq 4.$$

Hence Statement $2.r$ follows. Statement $3.r$ is vacuous since there are no vertices outside of W.

Next, we claim that Statements $1.(r+1)$, $2.(r+1)$ and $3.(r+1)$ imply Statements $1.r$ and $3.r$ for all $1 \leq r < |V|$.

To deduce Statement $1.r$, let us choose a sequence W of distinct vertices and a sequence L of not necessarily distinct indices such that the graph induced on W is connected $|W| = |L| = r$. Since $W \neq V$ there is a vertex v not in W with a neighbor in W, so that the graph induced on (W, v) is connected. Then by (7.1.9.1) we have

$$\mathrm{Hom}_L^W(Z) = \sum_{j=1}^{k} \mathrm{Hom}^{(W,v)}_{(L,j)}(Z). \tag{7.2.3.1}$$

By Statement $1.(r+1)$ we have $\mathrm{Hom}^{(W,v)}_{(L,j)}(Z) \neq 0$ for all $Z \in \mathcal{U}(\delta, \tau)$ and by Statement $2.(r+1)$ the angle between any two numbers

$$\mathrm{Hom}^{(W,v)}_{(L,i)}(Z) \neq 0 \quad \mathrm{Hom}^{(W,v)}_{(L,j)}(Z) \neq 0$$

does not exceed $\pi/2$. By Lemma 3.6.4 we have $\mathrm{Hom}_L^W(Z) \neq 0$ and hence Statement $1.r$ follows.

To deduce Statement $3.r$, let W and L be a sequence as above, $|W| = |L| = r$ and suppose that $W = (W', u)$ and $L = (L', l)$. Suppose that v is a neighbor of u

which is not in W and assume that $A, B \in \mathcal{U}(\delta, \tau)$ are two matrices that differ in the entries z_{lj}^{uv} for $j = 1, \ldots, k$ only. Let $C \in \mathcal{U}(\delta, \tau)$ be a matrix such that

$$c_{lj}^{uv} = 1 \quad \text{for} \quad j = 1, \ldots, k$$

and all other entries coincide with those in A and B. By Statement $1.(r+1)$ we have $\mathrm{Hom}_{(L,j)}^{(W,v)}(C) \neq 0$ for $j = 1, \ldots, k$ and by Statement $2.(r+1)$ the angle between any two complex numbers $\mathrm{Hom}_{(L,i)}^{(W,v)}(C) \neq 0$ and $\mathrm{Hom}_{(L,j)}^{(W,v)}(C) \neq 0$ does not exceed $\pi/2$. Applying (7.2.3.1), we can write

$$\mathrm{Hom}_L^W(A) = \sum_{j=1}^{k} a_{lj}^{uv} \mathrm{Hom}_{(L',l,j)}^{(W',u,v)}(C) \quad \text{and} \quad \mathrm{Hom}_L^W(B) = \sum_{j=1}^{k} b_{lj}^{uv} \mathrm{Hom}_{(L',l,j)}^{(W',u,v)}(C)$$

and by Lemma 3.7.3 the angle between

$$\mathrm{Hom}_L^W(A) \neq 0 \quad \text{and} \quad \mathrm{Hom}_L^W(B) \neq 0$$

does not exceed

$$2 \arctan \delta + 2 \arcsin \frac{\tau}{1-\delta},$$

which is equal to

$$\frac{\pi}{9} + \arctan \delta = \alpha \quad \text{if} \quad \Delta = 3$$

and is equal to

$$\frac{\pi}{4(\Delta - 1)} + \arctan \delta = \alpha \quad \text{if} \quad \Delta \geq 4.$$

Hence Statement $3.r$ follows.

Finally, we claim that Statements $1.(r+1)$, $2.(r+1)$ and $3.(r+1)$ imply Statement $2.r$ for $2 \leq r < |V|$. Let W be a sequence of distinct vertices such that the graph induced on W is connected, $|W| = r$ and $W = (W', u)$ and let L' be a sequence of not necessarily distinct indices such that $|L'| = r - 1$. Let $1 \leq l, m \leq k$ be any two indices. Given a matrix $A \in \mathcal{U}(\delta, \tau)$, we define a matrix $B \in \mathcal{U}(\delta, \tau)$ by

$$b_{lj}^{uv} = a_{mj}^{uv} \quad \text{for all} \quad v \quad \text{such that} \quad \{u, v\} \in E \quad \text{and all} \quad j = 1, \ldots, k \qquad (7.2.3.2)$$

and letting all other entries of B equal to the corresponding entries of A. Then

$$\mathrm{Hom}_{(L',l)}^{(W',u)}(B) = \mathrm{Hom}_{(L,m)}^{(W',u)}(A).$$

Let $d_0 \geq 1$ (we use that $r \geq 2$) be the number of neighbors v of u in the sequence W' and let $d_1 \leq \Delta - 1$ be the number of neighbors v of u not in W'. Then by Statement $1.r$ we have

$$\mathrm{Hom}_{(L',l)}^{(W',u)}(A) \neq 0 \quad \text{and} \quad \mathrm{Hom}_{(L',m)}^{(W',u)}(A) \neq 0$$

while by Statement 3.r the angle between non-zero complex numbers

$$\mathrm{Hom}_{(L',l)}^{(W',u)}(A) \quad \text{and} \quad \mathrm{Hom}_{(L,m)}^{(W',u)}(A) = \mathrm{Hom}_{(L',l)}^{(W',u)}(B)$$

does not exceed

$$2d_0 \arctan \frac{\tau}{1-\delta} + d_1\alpha. \tag{7.2.3.3}$$

If $\Delta = 3$ then (7.2.3.3) does not exceed

$$\frac{d_0\pi}{9} + \frac{d_1\pi}{9} + (d_1 - d_0) \arctan \delta. \tag{7.2.3.4}$$

If $d_1 \geq d_0$ then (7.2.3.4) does not exceed

$$\frac{2d_1\pi}{9} \leq \frac{4\pi}{9} < \frac{\pi}{2}$$

and if $d_1 < d_0$ then (7.2.3.4) does not exceed

$$\frac{d_0\pi}{9} + \frac{d_1\pi}{9} \leq \frac{\pi}{3} < \frac{\pi}{2}.$$

If $\Delta \geq 4$ then (7.2.3.3) does not exceed

$$\frac{d_0\pi}{4(\Delta-1)} + \frac{d_1\pi}{4(\Delta-1)} + (d_1 - d_0) \arctan \delta. \tag{7.2.3.5}$$

If $d_1 \geq d_0$ then (7.2.3.5) does not exceed

$$\frac{2d_1\pi}{4(\Delta-1)} \leq \frac{\pi}{2}$$

and if $d_1 < d_0$ then (7.2.3.5) does not exceed

$$\frac{\pi\Delta}{4(\Delta-1)} < \frac{\pi}{2}.$$

Hence Statement 2.r holds.

This proves Statements 1.1, 3.1 and 2.2. Let us choose a vertex u of the graph and two indices $1 \leq l, m \leq k$. Given a matrix $A \in \mathcal{U}(\delta, \tau)$, let us define a matrix B by (7.2.3.2). By Statement 1.1 we have $\mathrm{Hom}_l^u(A) \neq 0$ and $\mathrm{Hom}_l^u(B) \neq 0$ and by Statement 3.1 the angle between non-zero complex numbers $\mathrm{Hom}_l^u(A)$ and $\mathrm{Hom}_l^u(B) = \mathrm{Hom}_m^u(A)$ does not exceed

$$3\alpha \ < \ \frac{2\pi}{3} \quad \text{if} \quad \Delta = 3$$

and does not exceed

$$\alpha\Delta \ < \ \frac{\pi\Delta}{2\,(\Delta-1)} \ \leq \ \frac{2\pi}{3} \quad \text{if} \quad \Delta \geq 4.$$

By (7.1.9.1), we have

$$\mathrm{Hom}_G(Z) = \sum_{l=1}^{k} \mathrm{Hom}_l^u(Z)$$

and by Lemma 3.6.4, we have $\mathrm{Hom}_G(Z) \neq 0$ for all $Z \in \mathcal{U}\,(\delta, \tau)$. □

7.3 Graph Homomorphisms with Multiplicities

Following [BS16], we consider a refinement of the graph homomorphism partition function.

7.3.1 Definition. Let $G = (V, E)$ be an undirected graph with set V of vertices, set E of edges, without loops and multiple edges, and let $\Delta(G)$ be the largest degree of a vertex of G. We assume that $\Delta(G) \geq 1$. Let $m = (m_1, \ldots, m_k)$ be a vector of positive integers such that

$$m_1 + \cdots + m_k = |V|.$$

For a $k \times k$ symmetric complex matrix $A = \big(a_{ij}\big)$ we define the *partition function of graph homomorphisms with multiplicities m* by

$$\hom_{G,m}(A) = \sum_{\substack{\phi: V \longrightarrow \{1,\ldots,k\} \\ |\phi^{-1}(i)| = m_i \text{ for } i=1,\ldots,k}} \ \prod_{\{u,v\} \in E} a_{\phi(u)\phi(v)}. \tag{7.3.1.1}$$

Here the sum is taken over all maps $\phi : V \longrightarrow \{1, \ldots, k\}$ such that to every $i = 1, \ldots, k$ precisely m_i vertices are mapped. We observe that $\hom_{G,m}(A)$ is a polynomial in the entries a_{ij} of A and $\deg \hom_{G,m} = |E|$.

In [BS16], the following result is obtained.

7.3.2 Theorem. *There is an absolute constant $\delta_0 > 0$ (one can choose $\delta_0 = 0.108$) such that for every graph $G = (V, E)$ with the largest degree $\Delta(G) \geq 1$ of a vertex and every positive integer vector $m = (m_1, \ldots, m_k)$ of multiplicities such that $m_1 + \ldots + m_k = |V|$ we have*

$$\hom_{G,m}(A) \neq 0,$$

provided $A = (a_{ij})$ *is a* $k \times k$ *symmetric complex matrix satisfying*

$$|1 - a_{ij}| \;\le\; \frac{\delta_0}{\Delta(G)} \quad \textit{for all} \quad 1 \le i, j \le k.$$

As in Sect. 7.1, Theorem 7.3.2 implies the following corollary.

7.3.3 Theorem. *Let us fix some* $0 < \delta < \delta_0$*, where* δ_0 *is the constant in Theorem 7.3.2. Then there exists* $\gamma = \gamma(\delta_0/\delta) > 0$ *and for any* $0 < \epsilon < 1$*, any graph* $G = (V, E)$*, any positive integer* k*-vector* $m = (m_1, \ldots, m_k)$ *there is a polynomial* $p = p_{G,k,m,\delta,\epsilon}$ *in the entries of a* $k \times k$ *symmetric complex matrix* A *such that*

$$\deg p \;\le\; \gamma(\ln|E| - \ln\epsilon)$$

and

$$\left|\ln \hom_{G,m}(A) - p(A)\right| \;\le\; \epsilon$$

provided

$$|1 - a_{ij}| \;\le\; \frac{\delta}{\Delta(G)} \quad \textit{for all} \quad i, j.$$

As in Sect. 7.1, given G, m and ϵ, the polynomial p of Theorem 7.3.3 can be computed efficiently, in quasi-polynomial $(|E|k)^{O(\ln|E| - \ln\epsilon)}$ time. Given G, A and m, we define a univariate polynomial

$$g(z) = \hom_{G,m}\left(J + z(A - J)\right),$$

where $J = J_k$ is the $k \times k$ matrix filled by 1s, so that

$$g(0) = \hom_{G,m}(J) = \frac{|V|}{m_1! \cdots m_k!} \quad \text{and} \quad g(1) = \hom_{G,m}(A).$$

For an ordered set $I = (\{u_1, v_1\}, \ldots, \{u_s, v_s\})$ of distinct edges of G, let $V(I)$ be the set of vertices $\{u_1, v_1, \ldots, u_s, v_s\}$. Arguing as in Sect. 7.1,

$$\begin{aligned}
&\frac{d^s}{dz^s} g(z)\Big|_{z=0} \\
&\quad = \sum_{I=(\{u_1,v_1\},\ldots,\{u_s,v_s\})} \;\sum_{\substack{\phi: V(I) \longrightarrow \{1,\ldots,k\} \\ |\phi^{-1}(i)| \le m_i \text{ for } i=1,\ldots,k}} \frac{|V \setminus V(I)|!}{\big(m_1 - \phi^{-1}(1)\big)! \cdots \big(m_k - \phi^{-1}(k)\big)!} \\
&\qquad \times \big(a_{\phi(u_1)\phi(v_1)} - 1\big) \cdots \big(a_{\phi(u_s)\phi(v_s)} - 1\big).
\end{aligned}$$

Here the outer sum is taken over all ordered collections I of s edges of G, the inner sum is taken over all maps $\phi : V(I) \longrightarrow \{1, \ldots, k\}$ of the endpoints of the edges from I into the set $\{1, \ldots, k\}$ such that the inverse image of every $i \in \{1, \ldots, k\}$

consists of at most m_i points from $V(I)$. The multinomial coefficient

$$\frac{|V \setminus V(I)|!}{\big(m_1 - \phi^{-1}(1)\big)! \cdots \big(m_k - \phi^{-1}(k)\big)!}$$

accounts for the number of ways to extend ϕ to the whole set V of vertices of G in such a way that the inverse image of every $i \in \{1, \dots, k\}$ consists of exactly m_i points.

It follows that $g^{(s)}(0)$ is a polynomial of degree s in the entries of the matrix A computable in $(k|E|)^{O(s)}$ time. We define $f(z) = \ln g(z)$ and proceed as in Sect. 7.1 and in Sect. 3.6.7 before that.

We obtain a quasi-polynomial algorithm to approximate $\hom_{G,m}(A)$ within a given relative error ϵ, provided the matrix $A = \big(a_{ij}\big)$ satisfies $|1 - a_{ij}| \le \delta$ for all i, j and some fixed $0 < \delta < \delta_0$. Patel and Regts show [PR16] that the algorithm can be made genuinely polynomial provided $\Delta(G)$ is fixed in advance.

The functional $\hom_{G,m}(A)$ specializes to some combinatorial quantities of interest.

7.3.4 Hafnian. Suppose that G consists of n pairwise disjoint edges, so that $|V| = 2n$ and $\Delta(G) = 1$. Let $k = |V| = 2n$ and let $m = (1, \dots, 1)$. Then

$$\hom_{G,m}(A) = 2^n n! \operatorname{haf} A, \tag{7.3.4.1}$$

see Sect. 4.1.1. Indeed, every map $\phi : V \longrightarrow \{1, \dots, k\}$ in (7.3.1.1) is necessarily a bijection and the corresponding term is the product of weights $a_{i_1 j_1} \cdots a_{i_n j_n}$ in a perfect matching in the complete graph with $k = 2n$ vertices. Since $2^n n!$ different maps ϕ result in the same perfect matching (we can switch the vertices of each edge and also permute the edges), we obtain (7.3.4.1).

Theorem 4.1.5 is a particular case of Theorem 7.3.2 up to the value of δ_0, which is better in Theorem 4.1.5.

More generally, suppose that $k \ge 2n$, that $m = (1, \dots, 1)$ and that G consists of n pairwise disjoint edges and $k - 2n$ isolated points. Then

$$\hom_{G,m}(A) = (k - 2n)! 2^n n! \operatorname{haf}_n A,$$

where $\operatorname{haf}_n A$ enumerates matchings of size n in the complete graph with weights a_{ij} on the edges, see Sect. 5.1.1.

7.3.5 Hamiltonian permanent. Suppose that G is a cycle with n vertices, so that $|V| = n$ and $\Delta(G) = 2$. Let $k = |V| = n$ and let $m = (1, \dots, 1)$. Then

$$\hom_{G,m}(A) = n \operatorname{ham} A, \tag{7.3.5.1}$$

where $\operatorname{ham} A$ enumerates Hamiltonian cycles in the complete graph with n vertices and weights a_{ij} on the edges, see Sect. 3.8. The factor n in (7.3.5.1) accounts for

n distinct functions ϕ that differ by a cyclic shift of vertices of G and produce the same Hamiltonian cycle. It follows from Theorem 7.3.2 that ham $A \neq 0$ provided $A = (a_{ij})$ is a complex symmetric matrix satisfying $|1-a_{ij}| \leq \delta_0/2$ for all i, j where δ_0 is the constant in Theorem 7.3.2. Consequently, ham A can be approximated within relative error ϵ in quasi-polynomial time provided $|1 - a_{ij}| \leq \delta/2$, where $\delta < \delta_0$ is fixed in advance.

7.3.6 Enumerating independent sets. Let $k = 2$, let $m = (m_1, m_2)$ and let us choose

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

A map $\phi : V \longrightarrow \{1, 2\}$ contributes 1 to $\hom_{G,m}(A)$ in (7.3.1.1) if $\phi^{-1}(1) \subset V$ is an independent set and contributes 0 otherwise. Consequently, $\hom_{G,m}(A)$ is the number of independent sets in G of cardinality m_1. Detecting an independent set of a given size in a graph is a notoriously hard problem. For example, for any $0 < \epsilon < 1$ fixed in advance, it is an NP-hard problem to approximate the largest cardinality of an independent set in $G = (V, E)$ within a factor of $|V|^{1-\epsilon}$ [Hå99, Zu07].

Let us choose $0 < \delta < \delta_0$ as in Theorem 7.3.3 and let us define

$$\widehat{A} = \begin{pmatrix} 1 - \frac{\delta}{\Delta(G)} & 1 + \frac{\delta}{\Delta(G)} \\ 1 + \frac{\delta}{\Delta(G)} & 1 + \frac{\delta}{\Delta(G)} \end{pmatrix}.$$

Now $\hom_{G,m}(\widehat{A})$ can be approximated in quasi-polynomial time. For a subset $S \subset V$, let $e(S)$ be the number of edges of G spanned by the vertices of S. Then

$$\begin{aligned} \left(1 + \frac{\delta}{\Delta(G)}\right)^{-|E|} \hom_{G,m}(\widehat{A}) &= \sum_{\substack{S \subset V \\ |S| = m_1}} w(S) \quad \text{where} \\ w(S) &= \left(1 + \frac{\delta}{\Delta(G)}\right)^{-e(S)} \left(1 - \frac{\delta}{\Delta(G)}\right)^{e(S)}. \end{aligned} \tag{7.3.6.1}$$

In particular,

$$w(S) \;\leq\; \exp\left\{-2\delta \frac{e(S)}{\Delta(G)}\right\} \quad \text{and} \quad w(S) = 1 \quad \text{if } S \text{ is independent.}$$

Thus the sum (7.3.6.1) accounts for all subsets $S \subset V$ of m_1 vertices, where independent subsets are counted with weight 1, and all other subsets are counted with weight exponentially small in the number of edges they span. Computing (7.3.6.1) allows us to distinguish graphs that are sufficiently far from having an independent set of size m_1 (for example, when every subset in of m_1 vertices spans at least $\epsilon|E|$ edges for some $\epsilon > 0$) from graphs that have many independent sets of size m_1 (for example, when the probability that a randomly chosen m_1-subset is independent is

at least $2e^{-2\delta\epsilon|E|/\Delta(G)}$). Note that in the latter case, if G is not very far from regular, so that $|E|/\Delta(G)| \sim |V|$, the probability to hit such an independent set at random is exponentially small in $|V|$.

7.3.7 A multi-affine version of $\hom_{G,m}(A)$. We introduce an extension of $\hom_{G,m}(A)$. Let $Z = \left(z_{ij}^{uv}\right)$ be a $|E| \times \frac{k(k+1)}{2}$ matrix (tensor) indexed by edges $\{u, v\} \in E$ of the graph G and unordered pairs $\{i, j\}$ of not necessarily distinct indices $1 \le i, j \le k$. We write z_{ij}^{uv} instead of $z_{\{i,j\}}^{\{u,v\}}$ assuming that

$$z_{ij}^{uv} = z_{ij}^{vu} = z_{ji}^{vu} = z_{ji}^{uv}.$$

We define

$$\mathrm{Hom}_{G,m}(Z) = \sum_{\substack{\phi: V \longrightarrow \{1,\ldots,k\} \\ |\phi^{-1}(i)|=m_i \text{ for } i=1,\ldots,k}} \prod_{\{u,v\}\in E} z_{\phi(u)\phi(v)}^{uv}. \tag{7.3.7.1}$$

If $A = \left(a_{ij}\right)$ is $k \times k$ symmetric matrix and $z_{ij}^{uv} = a_{ij}$ for all $\{u, v\} \in E$, we clearly have $\hom_{G,m}(A) = \mathrm{Hom}_{G,m}(Z)$. The advantage of working with $\mathrm{Hom}_{G,m}(Z)$ is that it is *multi-affine*, that is, the degree of every variable in $\mathrm{Hom}_{G,m}(Z)$ does not exceed 1. We will prove that $\mathrm{Hom}_{G,m}(Z) \neq 0$ for complex $Z = \left(z_{ij}^{uv}\right)$ provided

$$\left|1 - z_{ij}^{uv}\right| \le \frac{\delta_0}{\Delta(G)} \quad \text{for all} \quad \{u, v\} \in E \quad \text{and all} \quad 1 \le i, j \le k,$$

where $\delta_0 > 0$ is an absolute constant (one can choose $\delta_0 = 0.108$).

Given $\delta > 0$, we define $\mathcal{U}(\delta) \subset \mathbb{C}^{|E|} \times \mathbb{C}^{k(k+1)/2}$,

$$\mathcal{U}(\delta) = \left\{Z = \left(z_{ij}^{uv}\right) : \quad \left|1 - z_{ij}^{uv}\right| \le \delta\right\} \tag{7.3.7.2}$$

(we suppress dependence on G in the notation). Hence our goal is to prove that

$$\mathrm{Hom}_{G,m}(Z) \neq 0 \quad \text{for all} \quad Z \in \mathcal{U}(\delta) \quad \text{where} \quad \delta = \frac{\delta_0}{\Delta(G)}.$$

7.3.8 Recursion. We need a version of the recurrence formula (7.1.9.1). Let $W = (v_1, \ldots, v_r)$ be an ordered sequence of vertices of G. A sequence W is called *admissible* if all vertices $v_1, \ldots, v_r$ are distinct. Let $L = (i_1, \ldots, i_r)$ be a sequence of indices $1 \le i_j \le k$. The *multiplicity* $m_i(L)$ of i in L is the number of occurrences of a given $1 \le i \le k$ in L:

$$m_i(L) = \left|j : \ i_j = i\right|.$$

We call a sequence L *admissible* if $m_i(L) \le m_i$ for $i = 1, \ldots, k$. For admissible sequences $W = (v_1, \ldots, v_r)$ of vertices and $L = (i_1, \ldots, i_r)$ of indices such that

$|W| = |L|$, we define

$$\mathrm{Hom}_L^W(Z) = \sum_{\substack{\phi:V\longrightarrow\{1,\dots,k\} \\ |\phi^{-1}(i)|=m_i \text{ for } i=1,\dots,k \\ \phi(v_j)=i_j \text{ for } j=1,\dots,r}} \prod_{\{u,v\}\in E} z^{uv}_{\phi(u)\phi(v)}.$$

Hence $\mathrm{Hom}_L^W(Z)$ is obtained by restricting the sum (7.3.1.1) to the maps ϕ that map the vertices v_j into i_j for $j = 1, \dots, r$. If $W = \emptyset$ and $L = \emptyset$ then $\mathrm{Hom}_L^W(Z) = \mathrm{Hom}_{G,m}(Z)$.

Let W be an admissible sequence of vertices, let L be an admissible sequence of indices such that $|W| = |L|$.

Let $v \in V$ be a vertex such that the sequence (W, v) obtained by appending W by v is admissible (that is, v is not in W). Then

$$\mathrm{Hom}_L^W(Z) = \sum_{\substack{i=1,\dots,k \\ (L,i) \text{ is admissible}}} \mathrm{Hom}_{(L,i)}^{(W,v)}(Z). \tag{7.3.8.1}$$

where (L, i) denotes the sequence L appended by i.

Let $1 \le i \le k$ be an index such that the sequence (L, i) is admissible. Then $m_i(L) < m_i$ and

$$\mathrm{Hom}_L^W(Z) = \frac{1}{m_i - m_i(L)} \sum_{\substack{v\in V \\ (W,v) \text{ is admissible}}} \mathrm{Hom}_{(L,i)}^{(W,v)}(Z). \tag{7.3.8.2}$$

We note that swapping the values on any two vertices $u, v \in V$ does not change the multiplicities of the values of ϕ.

To proceed with the induction, we need a simple geometric lemma which says that the sum of vectors rotates by a small angle if each vector is perturbed slightly and the vectors point roughly in the same direction.

7.3.9 Lemma. *Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be complex numbers such that $a_1, \dots, a_n$ are non-zero and*

$$\left|\frac{b_j}{a_j} - 1\right| \le \epsilon \quad \textit{for} \quad j = 1, \dots, n$$

and some $0 < \epsilon < 1$. Let

$$a = \sum_{j=1}^{n} a_j \quad \textit{and} \quad b = \sum_{j=1}^{n} b_j$$

and suppose that

$$|a| \;\geq\; \tau \sum_{j=1}^{n} |a_j|$$

for some $1 \geq \tau > \epsilon$. *Then* $a \neq 0$, $b \neq 0$ *and the angle between* a *and* b *does not exceed*

$$\arcsin \frac{\epsilon}{\tau}.$$

Proof. Clearly $a \neq 0$. Writing

$$b_j = (1+\epsilon_j)a_j \quad \text{where} \quad |\epsilon_j| \leq \epsilon \quad \text{for} \quad j = 1, \ldots, n,$$

we obtain

$$b = \sum_{j=1}^{n} (1+\epsilon_j)a_j = a + \sum_{j=1}^{n} \epsilon_j a_j \quad \text{where} \quad \left| \sum_{j=1}^{n} \epsilon_j a_j \right| \;\leq\; \epsilon \sum_{j=1}^{n} |a_j| \;\leq\; \frac{\epsilon}{\tau}|a|.$$

Hence

$$\left| \frac{b}{a} - 1 \right| \;\leq\; \frac{\epsilon}{\tau}$$

and

$$\left| \arg \frac{b}{a} \right| \;\leq\; \arcsin \frac{\epsilon}{\tau},$$

cf. Fig. 3.7. The proof now follows. □

Building on Lemma 7.3.9, we supply the first ingredient of our induction argument.

7.3.10 Lemma. *Let us fix an admissible sequence* W *of vertices, an admissible sequence* L *of indices such that* $0 \leq |W| = |L| \leq |V| - 2$, *a complex tensor* Z, *a real* $\epsilon > 0$ *and a real* $0 \leq \alpha < 2\pi/3$ *such that* $\epsilon < \cos(\alpha/2)$ *and let*

$$\omega = \arcsin \frac{\epsilon}{\cos(\alpha/2)}.$$

Suppose that for any two vertices $u, v \in V$ *and for any two indices* $1 \leq i, j \leq k$ *such that the sequences* (W, u, v) *and* (L, i, j) *are admissible, we have* $\mathrm{Hom}^{(W,u,v)}_{(L,i,j)}(Z) \neq 0$, $\mathrm{Hom}^{(W,u,v)}_{(L,j,i)}(Z) \neq 0$ *and*

$$\left| \frac{\mathrm{Hom}^{(W,u,v)}_{(L,j,i)}(Z)}{\mathrm{Hom}^{(W,u,v)}_{(L,i,j)}(Z)} - 1 \right| \;\leq\; \epsilon.$$

(1) *Let us fix two vertices* $v, u \in V$ *such that the sequence* (W, u, v) *is admissible and an index* i *such that the sequence* (L, i) *is admissible. Suppose that for any two indices* j_1 *and* j_2 *such that the sequences* (L, i, j_1) *and* (L, i, j_2) *are admissible, the angle between two non-zero complex numbers* $\mathrm{Hom}^{(W,u,v)}_{(L,i,j_1)}(Z)$ *and* $\mathrm{Hom}^{(W,u,v)}_{(L,i,j_2)}(Z)$ *does not exceed* α. *Then* $\mathrm{Hom}^{(W,u)}_{(L,i)}(Z) \neq 0$, $\mathrm{Hom}^{(W,v)}_{(L,i)}(Z) \neq 0$ *and the angle between the complex numbers does not exceed* ω.

(2) *Let us fix two indices* i *and* j, *possibly equal, such that the sequence* (L, i, j) *is admissible and a vertex* u *such that the sequence* (W, u) *is admissible. Suppose that for any two vertices* v_1 *and* v_2 *such that the sequences* (W, u, v_1) *and* (W, u, v_2) *are admissible, the angle between two non-zero complex numbers* $\mathrm{Hom}^{(W,u,v_1)}_{(L,i,j)}(Z)$ *and* $\mathrm{Hom}^{(W,u,v_2)}_{(L,i,j)}(Z)$ *does not exceed* α. *Then* $\mathrm{Hom}^{(W,u)}_{(L,i)}(Z) \neq 0$, $\mathrm{Hom}^{(W,u)}_{(L,j)}(Z) \neq 0$ *and the angle between the complex numbers does not exceed* ω.

Proof. To prove Part (1), using (7.3.8.1), we write

$$\mathrm{Hom}^{(W,u)}_{(L,i)}(Z) = \sum_{\substack{j=1,\dots,k \\ (L,i,j) \text{ is admissible}}} \mathrm{Hom}^{(W,u,v)}_{(L,i,j)}(Z) \quad \text{and}$$

$$\mathrm{Hom}^{(W,v)}_{(L,i)}(Z) = \sum_{\substack{j=1,\dots,k \\ (L,i,j) \text{ is admissible}}} \mathrm{Hom}^{(W,u,v)}_{(L,j,i)}(Z).$$

For j such that (L, i, j) is admissible, let us denote

$$a_j = \mathrm{Hom}^{(W,u,v)}_{(L,i,j)}(Z), \quad b_j = \mathrm{Hom}^{(W,u,v)}_{(L,j,i)}(Z),$$
$$a = \sum_j a_j \quad \text{and} \quad b = \sum_j b_j.$$

By Lemma 3.6.3, we have

$$|a| \geq \tau \sum_j |a_j| \quad \text{for} \quad \tau = \cos\frac{\alpha}{2}.$$

Since

$$a = \mathrm{Hom}^{(W,u)}_{(L,i)}(Z) \quad \text{and} \quad b = \mathrm{Hom}^{(W,v)}_{(L,i)}(Z),$$

the result follows by Lemma 7.3.9.

To prove Part (2), using (7.3.8.2), we write

$$\mathrm{Hom}^{(W,u)}_{(L,i)}(Z) = \frac{1}{m_j - m_j(L,i)} \sum_{\substack{v \in V \\ (W,u,v) \text{ is admissible}}} \mathrm{Hom}^{(W,u,v)}_{(L,i,j)}(Z) \quad \text{and}$$

$$\mathrm{Hom}^{(W,v)}_{(L,j)}(Z) = \frac{1}{m_i - m_i(L,\,j)} \sum_{\substack{u\in V\\ (W,u,v) \text{ is admissible}}} \mathrm{Hom}^{(W,u,v)}_{(L,j,i)}(Z).$$

For v such that (W, u, v) is admissible, let us denote

$$a_v = \mathrm{Hom}^{(W,u,v)}_{(L,i,j)}(Z), \quad b_v = \mathrm{Hom}^{(W,u,v)}_{(L,j,i)}(Z),$$
$$a = \sum_v a_v \quad \text{and} \quad b = \sum_v b_v.$$

By Lemma 3.6.3, we have

$$|a| \;\geq\; \tau \sum_v |a_v| \quad \text{for} \quad \tau = \cos\frac{\alpha}{2}.$$

By Lemma 7.3.9, the angle between non-zero complex numbers a and b does not exceed ω. Since

$$\mathrm{Hom}^{(W,u)}_{(L,i)}(Z) = \frac{1}{m_j - m_j(L,i)}a \quad \text{and} \quad \mathrm{Hom}^{(W,v)}_{(L,j)}(Z) = \frac{1}{m_i - m_i(L,\,j)}b,$$

the proof follows. □

7.3.11 Finding a fixed point. The gist of Lemma 7.3.10 is as follows. Suppose that the value of $\mathrm{Hom}^W_L(Z)$ does not change much if we permute any two indices in L, or, equivalently, any two vertices in W. We would like to know how the argument of the complex number $\mathrm{Hom}^W_L(Z)$ changes if we change one vertex in W or one index in L. Let $r = |W| = |L|$ be the length of the sequences. In Lemma 7.3.10 we show that if $\mathrm{Hom}^W_L(Z)$ does not rotate much when we change one index in L then $\mathrm{Hom}^{W'}_{L'}(Z)$ does not rotate much if we change one vertex in W' for *shorter* sequences $|W'| = |L'| = r - 1$ and if $\mathrm{Hom}^W_L(Z)$ does not rotate much when we change one vertex in W then $\mathrm{Hom}^{W'}_{L'}(Z)$ does not rotate much if we change one index in L' for *shorter* sequences $|W'| = |L'| = r - 1$.

We would like to find a fixed point of the conditions of Lemma 7.3.10 for which $\alpha = \omega$. That is, we want to find an $\epsilon > 0$ for which the equation

$$\alpha = \arcsin\frac{\epsilon}{\cos(\alpha/2)}$$

has a solution $0 \leq \alpha < 2\pi/3$. It is clear that for all sufficiently small $\epsilon > 0$ such a solution exists. In fact, any

$$0 < \epsilon \leq \max_{0\leq\alpha<2\pi/3}(\sin\alpha)\left(\cos\frac{\alpha}{2}\right) = \frac{4}{3\sqrt{3}} \tag{7.3.11.1}$$

will do.

Next, we link the property that $\mathrm{Hom}_L^W(Z)$ doesn't change much if any two vertices in W or any two indices in L are permuted with partial derivatives. We recall the definition (7.3.7.2) of the polydisc $\mathcal{U}(\delta)$.

7.3.12 Lemma. *Let us fix an integer $2 \le r \le |V|$ and real $\tau > 0$ and $0 < \delta < 1$. Suppose that for any admissible sequences W of vertices and L of indices such that $|W| = |L| = r$ and for any $Z \in \mathcal{U}(\delta)$ we have $\mathrm{Hom}_L^W(Z) \ne 0$ and the following condition holds: if $W = (W', v)$ and $L = (L', i)$, then*

$$\left|\mathrm{Hom}_L^W(Z)\right| \ \ge \ \frac{\tau}{\Delta(G)} \sum_{\substack{w:\ \{w,v\}\in E \\ l:\ 1\le l\le k}} \left|z_{il}^{vw}\right| \left|\frac{\partial}{\partial z_{il}^{vw}} \mathrm{Hom}_L^W(Z)\right|,$$

for any $Z \in \mathcal{U}(\delta)$.

Then for any admissible W and L such that $|W| = |L| = r$ and for any $Z \in \mathcal{U}(\delta)$, the following condition is satisfied: if $W = (W', u, v)$ and $L = (L', i, j)$ then

$$\left|\frac{\mathrm{Hom}_{(L',j,i)}^{(W',u,v)}(Z)}{\mathrm{Hom}_{(L',i,j)}^{(W',u,v)}(Z)} - 1\right| \ \le \ e^{\xi} - 1 \quad \textit{where} \quad \xi = \frac{4\delta\Delta(G)}{(1-\delta)\tau}.$$

Proof. Let us choose admissible W and L such that $|W| = |L| = r$ and suppose that $W = (W', u, v)$ and $L = (L', j, i)$. Without loss of generality, we assume that $i \ne j$. Since $\mathrm{Hom}_W^L(Z) \ne 0$ for all $Z \in \mathcal{U}(\delta)$, we choose a continuous branch of $\ln \mathrm{Hom}_L^W(Z)$, so that $\ln \mathrm{Hom}_L^W(Z)$ is real when Z is the matrix of 1s. Then

$$\frac{\partial}{\partial z_{il}^{vw}} \ln \mathrm{Hom}_L^W(Z) = \left(\frac{\partial}{\partial z_{il}^{vw}} \mathrm{Hom}_L^W(Z)\right) / \mathrm{Hom}_L^W(Z)$$

and using that the coordinates z_{ab}^{xy} of any $Z \in \mathcal{U}(\delta)$ satisfy $\left|z_{ab}^{xy}\right| \ge 1 - \delta$, we obtain

$$\begin{aligned} \sum_{\substack{w:\ \{w,v\}\in E \\ l:\ 1\le l\le k}} \left|\frac{\partial}{\partial z_{il}^{vw}} \ln \mathrm{Hom}_L^W(Z)\right| \ &\le \ \frac{\Delta(G)}{(1-\delta)\tau} \quad \text{and} \\ \sum_{\substack{w:\ \{w,u\}\in E \\ l:\ 1\le l\le k}} \left|\frac{\partial}{\partial z_{jl}^{uw}} \ln \mathrm{Hom}_L^W(Z)\right| \ &\le \ \frac{\Delta(G)}{(1-\delta)\tau}. \end{aligned} \tag{7.3.12.1}$$

Given a matrix $A \in \mathcal{U}(\delta)$, we define a matrix $B \in \mathcal{U}(\delta)$ by

$$\begin{aligned} b_{jl}^{uw} &= a_{il}^{uw} \quad \text{for all} \quad w \ne v \quad \text{such that} \quad \{u, w\} \in E \quad \text{and all} \quad l = 1, \dots, k \\ b_{il}^{vw} &= a_{jl}^{vw} \quad \text{for all} \quad w \ne u \quad \text{such that} \quad \{v, w\} \in E \quad \text{and all} \quad l = 1, \dots, k, \end{aligned}$$

while making all other entries of B equal to the corresponding entries of A. Then

$$\mathrm{Hom}^{(W',u,v)}_{(L',j,i)}(B) = \mathrm{Hom}^{(W',u,v)}_{(L',i,j)}(A)$$

and from (7.3.12.1), we conclude

$$\begin{aligned}
&\left|\ln \mathrm{Hom}^{(W',u,v)}_{(L',j,i)}(A) - \ln \mathrm{Hom}^{(W',u,v)}_{(L',i,j)}(A)\right| \\
&\quad = \left|\ln \mathrm{Hom}^{(W',u,v)}_{(L',j,i)}(A) - \ln \mathrm{Hom}^{(W',u,v)}_{(L',i,j)}(B)\right| \\
&\quad \le \max_{Z\in\mathcal{U}(\delta)} \left(\sum_{\substack{w:\ \{u,w\}\in E \\ l:\ 1\le l\le k}} \left| \frac{\partial}{\partial z^{uw}_{jl}} \ln \mathrm{Hom}^{W}_{L}(Z) \right| + \sum_{\substack{w:\ \{v,w\}\in E \\ l:\ 1\le l\le k}} \left| \frac{\partial}{\partial z^{vw}_{il}} \ln \mathrm{Hom}^{W}_{L}(Z) \right| \right) \\
&\quad \times \left(\max_{\substack{w\in W \\ 1\le l\le k}} \left| a^{uw}_{jl} - b^{uw}_{jl} \right|, \ \left| a^{uw}_{il} - b^{vw}_{il} \right| \right) \le \frac{2\Delta(G)}{(1-\delta)\tau} \times (2\delta) = \frac{4\delta\Delta(G)}{(1-\delta)\tau} = \xi.
\end{aligned}$$

Denoting

$$\zeta = \frac{\mathrm{Hom}^{(W',u,v)}_{(L',j,i)}(Z)}{\mathrm{Hom}^{(W',u,v)}_{(L',i,j)}(Z)},$$

we conclude that $|\ln \zeta| \le \xi$. Denoting $s = \ln \zeta$, we conclude

$$|\zeta - 1| = \left|e^s - 1\right| = \left|\sum_{n=1}^{\infty} \frac{s^n}{n!}\right| \ \le \ \sum_{n=1}^{\infty} \frac{|s|^n}{n!} = e^{\xi} - 1.$$

□

7.3.13 Tuning up ξ. We would like to have

$$e^{\xi} - 1 \ \le \ \epsilon$$

for some ϵ satisfying (7.3.11.1), see Sect. 7.3.11, so we choose

$$\xi = \ln(1+\epsilon).$$

Our next (and last) lemma relates the parameter τ in Lemma 7.3.12 to the angles between various complex numbers $\mathrm{Hom}^{W}_{L}(Z)$.

7.3.14 Lemma. *Let $0 \le \alpha < 2\pi/3$ be a real number, let W be an admissible sequence of vertices and let L be an admissible sequence of indices such that $1 \le |W| = |L| \le |V| - 1$. Suppose that for every $Z \in \mathcal{U}(\delta)$, for every w such that (W, w) is admissible and for every $1 \le l, j \le k$ such that (L, l) and (L, j) are admissible, we have $\mathrm{Hom}^{(W,w)}_{(L,l)}(Z) \ne 0$, $\mathrm{Hom}^{(W,w)}_{(L,j)}(Z) \ne 0$ and the angle between the two complex numbers does not exceed α.*

Suppose that $W = (W', v)$ and $L = (L', i)$. Then

$$\left|\mathrm{Hom}_L^W(Z)\right| \;\geq\; \frac{\tau}{\Delta(G)} \sum_{\substack{w:\ \{w,v\}\in E\\ j:\ 1\leq j\leq k}} \left|z_{ij}^{vw}\right| \left|\frac{\partial}{\partial z_{ij}^{vw}} \mathrm{Hom}_L^W(Z)\right| \quad \textit{for} \quad \tau = \cos\frac{\alpha}{2}.$$

Proof. Let w be a vertex such that $\{v, w\} \in E$. If w is an element of W' then

$$z_{ij}^{vw} \frac{\partial}{\partial z_{ij}^{vw}} \mathrm{Hom}_L^W(Z) = \begin{cases} \mathrm{Hom}_L^W(Z) & \text{if the element of } L' \text{ corresponding to } w \text{ is } j \\ 0 & \text{otherwise} \end{cases}$$

(here we use that $\mathrm{Hom}_L^W(Z)$ is a multi-affine function of Z).

If $w \neq v$ is not an element of W' then (W, w) is an admissible sequence of vertices and

$$z_{ij}^{vw} \frac{\partial}{\partial z_{ij}^{vw}} \mathrm{Hom}_L^W(Z) = \begin{cases} \mathrm{Hom}_{(L,j)}^{(W,w)}(Z) & \text{if } (L, j) \text{ is admissible} \\ 0 & \text{otherwise.} \end{cases}$$

By (7.3.8.1), if $w \neq v$ is not in W' then

$$\mathrm{Hom}_L^W(Z) = \sum_{\substack{j:\ 1\leq j\leq k\\ (L,j) \text{ is admissible}}} \mathrm{Hom}_{(L,j)}^{(W,w)}(Z)$$

and hence by Lemma 3.6.3,

$$\left|\mathrm{Hom}_L^W(Z)\right| \;\geq\; \tau \sum_{\substack{j:\ 1\leq j\leq k\\ (L,j) \text{ is admissible}}} \left|\mathrm{Hom}_{(L,j)}^{(W,w)}(Z)\right|.$$

Denoting by d_0 the number of vertices w such that $\{w, v\} \in E$ and w is an element of W' and by d_1 the number of vertices w such that $\{w, v\} \in E$ and w is not an element of W', we obtain

$$\begin{aligned} \sum_{\substack{w:\ \{w,v\}\in E\\ j:\ 1\leq j\leq k}} \left|z_{ij}^{vw}\right| \left|\frac{\partial}{\partial z_{ij}^{vw}} \mathrm{Hom}_L^W(Z)\right| &= d_0 \left|\mathrm{Hom}_L^W(Z)\right| \\ &+ \sum_{\substack{w:\ \{w,v\}\in E,\\ w \text{ is not in } W'\\ j:\ 1\leq j\leq k,\\ (L,j) \text{ is admissible}}} \left|\mathrm{Hom}_{(L,j)}^{(W,w)}(Z)\right| \\ &\leq d_0 \left|\mathrm{Hom}_L^W(Z)\right| + d_1\tau^{-1} \left|\mathrm{Hom}_L^W(Z)\right| \;\leq\; \frac{\Delta(G)}{\tau} \left|\mathrm{Hom}_L^W(Z)\right| \end{aligned}$$

and the proof follows. □

Now we are ready to prove Theorem 7.3.2.

7.3.15 Proof of Theorem 7.3.2. First we define some constants. For some $0 < \alpha < 2\pi/3$, to be specified later, we choose

$$\epsilon = (\sin\alpha)\left(\cos\frac{\alpha}{2}\right) \quad \text{so that} \quad \alpha = \arcsin\frac{\epsilon}{\cos(\alpha/2)},$$

see Sect. 7.3.11. Let

$$\xi = \ln(1+\epsilon) \quad \text{so that} \quad e^{\xi} - 1 = \epsilon,$$

see Sect. 7.3.13. and let

$$\tau = \cos\frac{\alpha}{2}.$$

see Lemma 7.3.14. We define

$$\delta_0 = \frac{\xi\tau}{4+\xi\tau}$$

and let

$$\delta = \frac{\delta_0}{\Delta(G)},$$

so that

$$\frac{4\delta\Delta(G)}{(1-\delta)\tau} \;\le\; \xi,$$

see Lemma 7.3.12. As our goal is to maximize δ_0, we choose α to maximize

$$\xi\tau = \left(\cos\frac{\alpha}{2}\right)\ln\left(1+(\sin\alpha)\left(\cos\frac{\alpha}{2}\right)\right).$$

Numerical computations show that it is reasonable to choose

$$\alpha = 1$$

so that

$$\epsilon \approx 0.74, \quad \xi \approx 0.55, \quad \tau \approx 0.88$$

and

$$\delta_0 > 0.108.$$

Our goal is to show that

$$\mathrm{Hom}_{G,m}(Z) \neq 0 \quad \text{for all} \quad Z \in \mathcal{U}(\delta).$$

We prove by descending induction for $r = |V|, |V| - 1, \ldots, 2$ the following Statements 1.r–5.r.

Statement 1.r. Let W be an admissible sequence of vertices and let L be an admissible sequence of indices such that $|W| = |L| = r$. Then $\mathrm{Hom}_L^W(Z) \neq 0$.

Statement 2.r. Let W be an admissible sequence of vertices and let L be an admissible sequence of indices such that $|W| = |L| = r$. If $W = (W', v)$ and $L = (L', i)$ then

$$\left|\mathrm{Hom}_L^W(Z)\right| \;\geq\; \frac{\tau}{\Delta(G)} \sum_{\substack{w:\ \{w,v\}\in E \\ l:\ 1\leq l\leq k}} \left|z_{il}^{vw}\right| \left|\frac{\partial}{\partial z_{il}^{vw}} \mathrm{Hom}_L^W(Z)\right|.$$

Statement 3.r. Let W be an admissible sequence of vertices and let L be an admissible sequence of indices such that $|W| = |L| = r$. If $W = (W', u, v)$ and $L = (L', i, j)$ then

$$\left|\frac{\mathrm{Hom}_{(L',j,i)}^{(W',u,v)}(Z)}{\mathrm{Hom}_{(L',i,j)}^{(W',u,v)}(Z)} - 1\right| \;\leq\; \epsilon.$$

Statement 4.r. Let W be an admissible sequence of vertices such that $|W| = r$. Suppose that $W = (W', w)$ and let L' be an admissible sequence of indices such that $|L'| = r - 1$. Let i and j be indices such that the sequences (L', i) and (L', j) are admissible. Then $\mathrm{Hom}_{(L',i)}^{(W',w)}(Z) \neq 0$, $\mathrm{Hom}_{(L',j)}^{(W',w)}(Z) \neq 0$ and the angle between the complex numbers does not exceed α.

Statement 5.r. Let L be an admissible sequence of vertices such that $|L| = r$. Suppose that $L = (L', i)$ and let W' be an admissible sequence of vertices such that $|W'| = r - 1$. Let u and v be vertices such that the sequences (W, u) and (W, v) are admissible. Then $\mathrm{Hom}_{(L',i)}^{(W',u)}(Z) \neq 0$, $\mathrm{Hom}_{(L',i)}^{(W',v)}(Z) \neq 0$ and the angle between the complex numbers does not exceed α.

Suppose that $r = |V|$ and let $W = (v_1, \ldots, v_r)$ and $L = (i_1, \ldots, i_r)$. Then

$$\mathrm{Hom}_L^W(Z) = \prod_{\substack{1\leq j<l\leq r: \\ \{v_j,v_l\}\in E}} z_{i_j i_l}^{v_j v_l},$$

and hence Statement 1.r holds. Furthermore, if $\deg v_r$ is the degree of v_r, we get

$$\sum_{\substack{w:\ \{w,v_r\}\in E \\ l:\ 1\leq l\leq k}} \left|z_{i_r l}^{v_r w}\right| \left|\frac{\partial}{\partial z_{i_r l}^{v_r w}} \mathrm{Hom}_L^W(Z)\right| = (\deg v_r) \left|\mathrm{Hom}_L^W(Z)\right|$$

and Statement 2.r follows as well. Lemma 7.3.12 implies that Statement 3.r holds. Statements 4.r and 5.r hold since if L' is an admissible sequence of indices such that $|L'| = |V| - 1$ then there is a unique index i such that the sequence (L', i) is admissible and if W' is an admissible sequence of vertices such that $|W'| = |V| - 1$ then there is a unique vertex w such that the sequence (W', w) is admissible.

From formula (7.3.8.1) and Lemma 3.6.3, we get the implication:

$$\text{Statement } 1.\, r \quad \text{and} \quad \text{Statement } 4.r \Longrightarrow \text{Statement } 1.(r-1).$$

From Lemma 7.3.14, we get the implication

$$\text{Statement } 4.r \Longrightarrow \text{Statement } 2.(r-1).$$

From Lemma 7.3.12, we get the implication

$$\text{Statement } 1.(r-1) \quad \text{and} \quad \text{Statement } 2.(r-1) \Longrightarrow \text{Statement } 3.(r-1).$$

From Part 1 of Lemma 7.3.10, we get the implication

$$\text{Statement } 3.r \quad \text{and} \quad \text{Statement } 4.r \Longrightarrow \text{Statement } 5.(r-1).$$

From Part 2 of Lemma 7.3.10, we get the implication

$$\text{Statement } 3.r \quad \text{and} \quad \text{Statement } 5.r \Longrightarrow \text{Statement } 4.(r-1).$$

This proves Statements 1.2–5.2. Applying again Part 2 of Lemma 7.3.10, we get the implication

$$\text{Statement } 3.2 \quad \text{and} \quad \text{Statement } 5.2 \Longrightarrow \text{Statement } 4.1.$$

Then from formula (7.3.8.1) and Lemma 3.6.3, we get the implication

$$\text{Statement } 4.1 \Longrightarrow \text{Statement } 1.0,$$

which completes the proof. □

7.4 The Lee–Yang Circle Theorem and the Ising Model

Our goal is to prove the following remarkable theorem of Lee and Yang [LY52].

7.4.1 Theorem. *Let $A = \left(a_{ij}\right)$ be an $n \times n$ complex Hermitian matrix (thus we have $a_{ij} = \overline{a_{ji}}$ for all i, j) such that $|a_{ij}| \leq 1$ for all $1 \leq i, j \leq n$. Let us define a univariate polynomial*

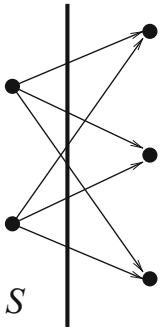


Fig. 7.2 The cut created by a set S of vertices and the directed edges contributing to the weight of the cut

$$\mathrm{Cut}_A(z) = \sum_{S \subset \{1,\dots,n\}} z^{|S|} \prod_{\substack{i \in S \\ j \notin S}} a_{ij}.$$

Then every root z_0 *of* Cut_A *satisfies* $|z_0| = 1$.

The polynomial $\mathrm{Cut}_A(z)$ enumerates all cuts in the complete directed graph G with set $\{1, \dots, n\}$ of vertices and weight a_{ij} on the edge $i \to j$, cf. Sect. 7.1.11. Every subset $S \subset \{1, \dots, n\}$ of vertices, including $S = \emptyset$, creates a cut. The weight of the cut is the product of weights of all directed edges of G that originate in S and end outside of S (for $S = \emptyset$ and for $S = \{1, \dots, n\}$ the weight of the cut is 1), see Fig. 7.2, while the monomial $z^{|S|}$ accounts for the cardinality of the set S.

We note that the weights of the cuts corresponding to a set S and to its complement are complex conjugates of each other and hence

$$z^n \,\mathrm{Cut}_A\left(\frac{1}{z}\right) = \overline{\mathrm{Cut}_A(\overline{z})}.$$

As follows from Lemma 2.2.1, see also Sect. 3.6, Theorem 4.1.5, Theorem 4.4.2, Sect. 6.1.5, Theorems 7.1.5 and 7.2.3, for any $0 < \delta < 1$, fixed in advance and any $0 < \epsilon < 1$ there is a polynomial $p = p_{n,\delta,\epsilon}$ in z and the entries a_{ij} of an $n \times n$ Hermitian matrix $A = \big(a_{ij}\big)$ such that $\deg p = O\,(\ln n - \ln \epsilon)$ and

$$|\ln \mathrm{Cut}_A(z) - p(A, z)| \;\le\; \epsilon$$

provided $|a_{ij}| \le 1$ for all i, j and $|z| \le \delta$. As before, the approximating polynomial p can be computed in $n^{O(\ln n - \ln \epsilon)}$ time.

Our proof follows [Hi97], see also [Ru71] and [As70].

7.4.2 Lemma. *Let* a *be a complex number such that* $|a| \le 1$ *and let* z_1 *and* z_2 *be complex numbers such that* $|z_1|, |z_2| < 1$. *Then*

$$1 + az_1 + \overline{a}z_2 + z_1 z_2 \neq 0.$$

Proof. If $|a| = 1$ then $a\overline{a} = 1$ and

$$1 + az_1 + \overline{a}z_2 + z_1z_2 = (1 + az_1)\,(1 + \overline{a}z_2)$$

and the proof follows. Hence we may assume that $|a| < 1$. Solving the equation

$$1 + az_1 + \overline{a}z_2 + z_1z_2 = 0$$

for z_2, we obtain

$$z_2 = -\frac{1 + az_1}{\overline{a} + z_1}. \tag{7.4.2.1}$$

For any z such that $|z| = 1$, we have

$$|1 + az| = |1 + \overline{a}\overline{z}| \quad \text{and} \quad |\overline{a} + z| = |\overline{a} + z||\overline{z}| = |\overline{a}z + 1|,$$

from which it follows that the transformation

$$z \longmapsto -\frac{1 + az}{\overline{a} + z}$$

maps the unit circle $|z| = 1$ onto itself and the disc $|z| < 1$ onto its complement $|z| > 1$ (we use that $|a| < 1$). Therefore, if z_2 satisfies (7.4.2.1) with some $|z_1| < 1$, we must have $|z_2| > 1$ and the proof follows. □

7.4.3 Proof of Theorem 7.4.1. Let us consider an n-variate polynomial

$$p_A(z_1, \ldots, z_n) = \sum_{S \subset \{1,\ldots,n\}} \mathbf{z}^S \prod_{\substack{i \in S \\ j \notin S}} a_{ij}, \quad \text{where} \quad \mathbf{z}^S = \prod_{i \in S} z_i. \tag{7.4.3.1}$$

For $1 \le i < j \le n$, let us define

$$\begin{aligned} p_{ij}(z_1, \ldots, z_n) &= \sum_{\substack{S \subset \{1,\ldots,n\} \\ i \notin S,\, j \notin S}} \mathbf{z}^S + \sum_{\substack{S \subset \{1,\ldots,n\} \\ i \in S,\, j \in S}} \mathbf{z}^S + a_{ij} \sum_{\substack{S \subset \{1,\ldots,n\} \\ i \in S,\, j \notin S}} \mathbf{z}^S + a_{ji} \sum_{\substack{S \subset \{1,\ldots,n\} \\ i \notin S,\, j \in S}} \mathbf{z}^S \\ &= \left(1 + a_{ij}z_i + a_{ji}z_j + z_iz_j\right) \left(\sum_{\substack{S \subset \{1,\ldots,n\} \\ i \notin S,\, j \notin S}} \mathbf{z}^S \right) \\ &= \left(1 + a_{ij}z_i + a_{ji}z_j + z_iz_j\right) \prod_{k \in \{1,\ldots,n\} \setminus \{i,j\}} (1 + z_k)\,. \end{aligned}$$

From Lemma 7.4.2 it follows that

$$p_{ij}(z_1, \ldots, z_n) \neq 0 \quad \text{provided} \quad |z_1|, \ldots, |z_n| < 1.$$

Therefore, for any real $0 < \rho < 1$, the polynomial

$$(z_1, \ldots, z_n) \longmapsto p_{ij}(\rho z_1, \ldots, \rho z_n)$$

is $\mathbb{D}$-stable, see Sect. 2.5. On the other hand, p_A is the Schur (Hadamard) product of the polynomials p_{ij} over all pairs $1 \le i < j \le n$. Therefore, by Theorem 2.5.1, for any real $0 < \rho < 1$, the polynomial

$$\sum_{S \subset \{1,\ldots,n\}} \mathbf{z}^S \rho^{|S|n(n-1)/2} \prod_{\substack{i \in S \\ j \notin S}} a_{ij}$$

is $\mathbb{D}$-stable. Taking the limit as $\rho \longrightarrow 1$, by Hurwitz's Theorem, cf. the proof of Lemma 2.4.2, we conclude that

$$p_A(z_1, \ldots, z_n) \neq 0 \quad \text{provided} \quad |z_1|, \ldots, |z_n| < 1.$$

Therefore,

$$\mathrm{Cut}_A(z) = p_A(z, \ldots, z) \neq 0 \quad \text{provided} \quad |z| < 1.$$

Since

$$z^n \mathrm{Cut}_A\left(\frac{1}{z}\right) = \overline{\mathrm{Cut}_A(\overline{z})},$$

we conclude that

$$\mathrm{Cut}_A(z) \neq 0 \quad \text{provided} \quad |z| > 1$$

and the proof follows. □

As follows from our proof, we have

$$p_A(z_1, \ldots, z_n) = \sum_{S \subset \{1,\ldots,n\}} \mathbf{z}^S \prod_{\substack{i \in S \\ j \notin S}} a_{ij} \neq 0$$

provided

$$|z_i| < 1 \quad \text{for} \quad i = 1, \ldots, n.$$

Consequently, for any $0 < \delta < 1$, fixed in advance, there is an algorithm which, given a Hermitian matrix $A = (a_{ij})$ such that $|a_{ij}| \le 1$ for all i and j, complex $z_1, \ldots, z_n$ such that $|z_i| \le \delta$ for $i = 1, \ldots, n$ and a real $0 < \epsilon < 1$ approximates $p_A(z_1, \ldots, z_n)$ within a relative error of ϵ in $n^{O(\ln n - \ln \epsilon)}$ time. For a Markov Chain Monte Carlo based algorithm, see [JS93].

7.4.4 The Ising model. One of the oldest and most famous models in statistical physics, the *Ising model*, seeks to explain the phase transition in magnetization. It is described as follows: let $G = (V, E)$ be an undirected graph without loops or

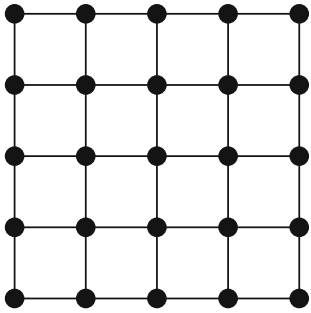


Fig. 7.3 The graph of a rectangular region of $\mathbb{Z}^2$

multiple edges. Typically, G is a graph of a rectangular region of the 2-dimensional integer grid $\mathbb{Z}^2$ or a cubical region of the 3-dimensional grid $\mathbb{Z}^3$, see Fig. 7.3.

We think of the vertices of G, which we number $1, 2, \ldots, |V|$, as of *atoms*. Suppose that some real numbers b_{ij} for $\{i, j\} \in E$ are attached to the edges of G, which characterize interactions between neighboring atoms and that real numbers c_i for $i = 1, \ldots, N$ are attached to the vertices of G, which characterize the external magnetic field. An assignment $\sigma : V \longrightarrow \{-1, 1\}$ of signs to the vertices of G is called a *configuration* and the signs themselves are interpreted as *spins* of the atoms. The *energy* of the configuration σ is defined as

$$H(\sigma) = -\sum_{\{i,j\}\in E} b_{ij}\sigma(i)\sigma(j) - \sum_{i\in V} c_i\sigma(i).$$

The *partition function* of the Ising model is just the sum over all $2^{|V|}$ configurations:

$$\begin{aligned} Z(G,t) &= \sum_{\sigma:V\to\{-1,1\}} e^{-\gamma H(\sigma)/t} \\ &= \sum_{\sigma:V\longrightarrow\{-1,1\}} \exp\left\{\gamma t^{-1}\left(\sum_{\{i,j\}\in E} b_{ij}\sigma(i)\sigma(j) + \sum_{i\in V} c_i\sigma(i)\right)\right\}, \end{aligned} \tag{7.4.4.1}$$

where $t > 0$ is a parameter interpreted as the *temperature* and $\gamma > 0$ is an absolute constant. The partition function defines a probability distribution on the set of all $2^{|V|}$ configurations:

$$\mathbf{Pr}(\sigma) = \frac{e^{-\gamma H(\sigma)/t}}{Z(G,t)} \quad \text{for} \quad \sigma : V \longrightarrow \{-1, 1\}. \tag{7.4.4.2}$$

Some observations are in order. As the temperature $t \longrightarrow +\infty$ grows, the distribution approaches the uniform distribution on the set of all configurations. As the temperature $t \longrightarrow 0+$ falls to 0, the distribution concentrates on the configurations with the lowest energy. Suppose that $c_i = 0$ for all i, so that there is no external

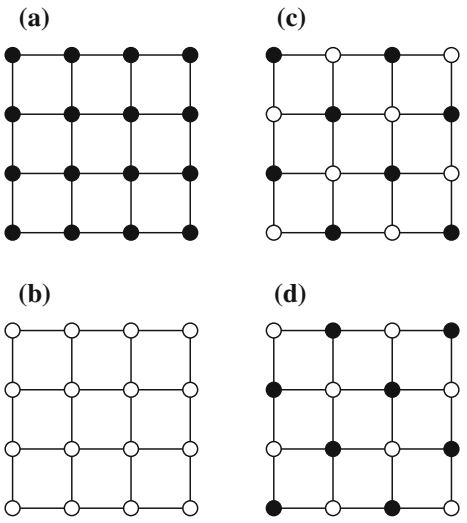


Fig. 7.4 The most likely configurations in the ferromagnetic case **a** and **b** and in the anti-ferromagnetic case **c** and **d**. *Black dots* denote spins $+1$ while *white dots* denote spins -1

magnetic field. If all $b_{ij} > 0$ then the configurations where the spins of neighboring atoms coincide have lower energy and hence higher probability. This is called the *ferromagnetic* case. If all $b_{ij} < 0$ then the configurations where the spins of neighboring atoms are opposite have lower energy and hence higher probability. This is called the *anti-ferromagnetic* case, see Fig. 7.4.

One can observe now that by a change of variables, $Z(G,t)$ is transformed into the partition function of Theorem 7.4.1, more precisely into (7.4.3.1). Namely, we write

$$Z(G,t) = \exp\left\{\gamma t^{-1}\left(\sum_{\{i,j\}\in E} b_{ij} + \sum_{i\in V} c_i\right)\right\} \times \sum_{\sigma\colon V\to\{-1,1\}} \exp\left\{-2\gamma t^{-1}\left(\sum_{\substack{\{i,j\}\in E:\\ \sigma(i)\neq\sigma(j)}} b_{ij} + \sum_{\substack{i\in V:\\ \sigma(i)=-1}} c_i\right)\right\}.$$

A configuration $\sigma : V \longrightarrow \{-1,1\}$ is uniquely determined by the subset $S \subset V$ of vertices where $\sigma(i) = 1$. Hence letting

$$a_{ij} = \exp\left\{-2\gamma t^{-1} b_{ij}\right\} \quad \text{and} \quad z_i = \exp\left\{-2\gamma t^{-1} c_i\right\},$$

we can further write

$$Z(G,t) = \exp\left\{\gamma t^{-1}\left(\sum_{\{i,j\}\in E} b_{ij} + \sum_{i\in V} c_i\right)\right\} \sum_{S\subset V} \mathbf{z}^S \prod_{\substack{i\in S\\ j\notin S}} a_{ij},$$

where we agree that $a_{ij} = 1$ if $\{i, j\} \notin E$ (equivalently, we agree that $b_{ij} = 0$ if $\{i, j\} \notin E$) and

$$\mathbf{z}^S = \prod_{i\in S} z_i.$$

Hence up to a simple factor, $Z(G, t)$ is indeed transformed into the partition function (7.4.3.1) of cuts. Moreover, the case of $|a_{ij}| \leq 1$ treated by Theorem 7.4.1 corresponds to the ferromagnetic case of $b_{ij} \geq 0$. Theorem 7.4.1 thus says that in the ferromagnetic case the roots of $c \longmapsto Z(G, t; c)$, as a function of the constant magnetic field $c_i = c$ interpreted as a complex variable, are *purely imaginary*, that is, satisfy $\Re\, c = 0$.

There are two related, though not identical, notions of a phase transition in the Ising model. Both are asymptotic, as the graph grows in some regular way (for example, when the square region in $\mathbb{Z}^2$ or the cubical region in $\mathbb{Z}^3$ gets larger, see Fig. 7.3). The first notion has to do with complex zeros of the partition function $Z(G, t)$ defined by (7.4.4.1). Various quantities that have physical interpretation can be expressed in terms of the "free energy per atom"

$$\frac{1}{|V|} \ln Z(G,t),$$

see [Ci87] and references therein. If for a sequence $G_n = (V_n, E_n)$ of growing graphs a complex zero of the function $t \longmapsto Z(G_n, t)$ approaches the positive real axis, it means that the "thermodynamic limit"

$$\lim_{n\longrightarrow\infty} \frac{1}{|V_n|} \ln Z(G_n,t)$$

hits a singularity at some temperature t_c, and hence those physical quantities hit a singularity (discontinuity or loss of smoothness) as well, which is an indication of a phase transition (such as the loss of the magnetization or gas becoming liquid, etc.) occurring at the temperature t_c, see [YL52]. Hence Theorem 7.4.1 implies that as long as the magnetic field remains constant and non-zero, there is no phase transition in the ferromagnetic case, as all the zeros of $t \longmapsto Z(G_n, t)$ stay away from the positive real axis even as the graph G_n grows. If the external magnetic field is zero, Onsager [On44] demonstrated that there is indeed a phase transition at a particular temperature in the case of a growing rectangular region of the 2-dimensional grid as on Fig. 7.3, in the ferromagnetic case with constant interactions $b_{ij} > 0$, see also Chap. 10 of [Ai07] for the computation of the partition function in that case.

The second notion of the phase transition has to do with the correlation decay phenomenon, as in Sects. 5.2, 6.3 and 6.4. Suppose that there is no external magnetic

field, so that $c_i = 0$ for all $i \in V$. Let us consider the probability distribution on the set of configurations σ defined by (7.4.4.2). We choose a particular vertex i in G and consider the conditional probability that $\sigma(i) = 1$, given that the spins of the vertices far away from i are also equal to 1. For example, i is the central vertex on Fig. 7.3, the spins are fixed to 1 on the boundary of the square and the size of the square is allowed to grow. We say that the phase transition occurs at a particular temperature t_c if for higher temperatures $t > t_c$ the probability that $\sigma(i) = 1$ asymptotically does not depend on the boundary conditions (no long range interactions), while for lower temperatures $t < t_c$ the probability that $\sigma(i) = 1$ asymptotically depends on the boundary conditions (long range interactions appear). In 1936, Peierls found a relatively simple argument, which allows one to show that this kind of the phase transition indeed occurs for a variety of graphs, in particular for grids in $\mathbb{Z}^d$ with $d \geq 2$, see [Ci87] for an exposition and references.

7.4.5 Reduction to matchings. Fisher [Fi66] showed that computing the partition function $Z(G, t)$ defined by (7.4.4.1) in the case of zero magnetic field (that is, then $c_i = 0$ for all $i \in V$) can be reduced to counting weighted perfect matchings in some auxiliary graph $\widehat{G}$, that is, to computing an appropriate hafnian, see Sect. 4.1. Moreover, if G is a planar graph, the graph $\widehat{G}$ is also planar, so one can use Pfaffians to compute $Z(G, t)$, see Sect. 4.3. Heilmann and Lieb [HL72] modified Fisher's argument to account for a non-zero magnetic field and showed that in general computing $Z(G, t)$ reduces to computing the matching polynomial of a graph, see Chap. 5. Below we follow [Fi66].

To simplify the notation, we write (7.4.4.1) in the absence of the magnetic field simply as

$$Z(G) = \sum_{\sigma: V \to \{-1,1\}} \prod_{\{i,j\} \in E} \exp\bigl\{b_{ij}\sigma(i)\sigma(j)\bigr\},$$

where b_{ij} are some real weights on the edges of E. Since the product $\sigma(i)\sigma(j)$ takes only two values, $+1$ and -1, we can interpolate $\exp\bigl\{b_{ij}\sigma(i)\sigma(j)\bigr\}$ by an affine function in $\sigma(i)\sigma(j)$ and write

$$Z(G) = \sum_{\sigma: V \to \{-1,1\}} \prod_{\{i,j\} \in E} \bigl(f_{ij} + g_{ij}\sigma(i)\sigma(j)\bigr)$$
$$\text{where} \quad f_{ij} = \frac{e^{b_{ij}} + e^{-b_{ij}}}{2} > 0 \quad \text{and} \quad g_{ij} = \frac{e^{b_{ij}} - e^{-b_{ij}}}{2}.$$

Next, we factor out f_{ij} and write

$$\begin{aligned} Z(G) &= \left(\prod_{\{i,j\} \in E} f_{ij}\right) Z_0(G) \quad \text{where} \\ Z_0(G) &= \sum_{\sigma: V \to \{-1,1\}} \prod_{\{i,j\} \in E} \bigl(1 + h_{ij}\sigma(i)\sigma(j)\bigr) \quad \text{and} \quad h_{ij} = \frac{e^{b_{ij}} - e^{-b_{ij}}}{e^{b_{ij}} + e^{-b_{ij}}}. \end{aligned} \tag{7.4.5.1}$$

We note that the signs of h_{ij} and b_{ij} coincide. We will be computing $Z_0(G)$.

Let us expand the product in the definition of $Z_0(G)$. We obtain various monomials of the type

$$h_{i_1 j_1} \cdots h_{i_s j_s} \sigma(i_1)\sigma(j_1) \cdots \sigma(i_s)\sigma(j_s). \tag{7.4.5.2}$$

The monomials that survive summing over all $\sigma : V \longrightarrow \{-1, 1\}$ correspond to the collections T of distinct edges $\{i_1, j_1\}, \ldots, \{i_s, j_s\}$ that cover every vertex i of V an even, possibly zero, number of times. We call such collections *Eulerian*. Hence we can write

$$Z_0(G) = 2^{|V|} \sum_{\substack{T \subset E \\ T \text{ is Eulerian}}} \prod_{\{i,j\} \in T} h_{ij}. \tag{7.4.5.3}$$

Next, we begin to modify G. First, we construct an intermediate graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with weights on edges such that $Z_0(G) = Z_0(\tilde{G})$ and the degree of every vertex on $\tilde{G}$ does not exceed 3. We do it step by step, each time replacing a vertex of degree $d > 3$ by d clones, connected in a circular order as on Fig. 7.5.

The edges of the obtained graph are of the two kinds: the *inherited* edges, connecting clones of the vertex to other vertices (thick lines on Fig. 7.5) and *circular* edges, connecting clones of the vertex of G within themselves (thin lines on Fig. 7.5). The weights $\tilde{h}_{ij}$ on the inherited edges are copied from those on the corresponding edges of G, while for any circular edge $\{i, j\}$ we let $\tilde{h}_{ij} = 1$. Whenever $\sigma(i) \neq \sigma(j)$ for some two clones of a vertex in G, we also have $\sigma(i) \neq \sigma(j)$ for two neighboring clones and hence by (7.4.5.1) the contribution of the corresponding configuration σ to the partition function is just 0. Repeating this process, we obtain a graph $\tilde{G}$ with vertices of degree 1, 2 and 3 and such that $Z_0(G) = Z_0(\tilde{G})$. Note that if G is planar then $\tilde{G}$ is also planar.

Hence without loss of generality, we may assume the degree of every vertex of G is 1, 2 or 3. We still denote weights on the edges of G by h_{ij} and without loss of generality we assume that $h_{ij} > 0$. Next, we construct a weighted graph $\widehat{G}$ such that $Z_0(G)$ is expressed as the partition function enumerating perfect matchings in $\widehat{G}$, see Sect. 4.1.

We keep vertices of degree 1 intact. If h_{ij} is the weight on the unique edge incident to such a vertex, we assign weight $w_{ij} = 1/h_{ij}$ to the unique *inherited* edge in $\widehat{G}$

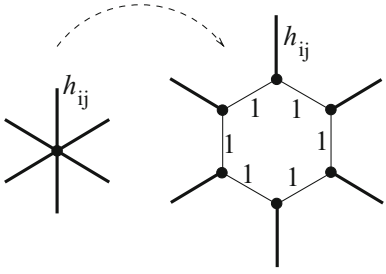


Fig. 7.5 Replacing a vertex of degree 6 by 6 vertices of degree 3

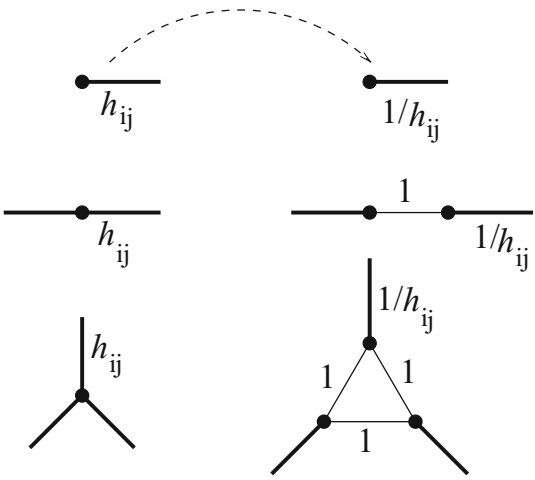


Fig. 7.6 Constructing $\widehat{G}$ from G

incident to the vertex, see Fig. 7.6. Every vertex of G of degree 2 we replace by two clones, connected by an *auxiliary* edge of weight 1 in $\widehat{G}$ (thin line on Fig. 7.6) and connected to other vertices by two *inherited* edges (thick lines on Fig. 7.6). If an edge incident in G to such a vertex has weight h_{ij}, we assign weight $w_{ij} = 1/h_{ij}$ to the corresponding inherited edge in $\widehat{G}$. Every vertex G of degree 3 we replace by three clones connected by *auxiliary* edges of weight 1 each (thin lines on Fig. 7.6) and connected to other vertices by three *inherited* edges (thick lines on Fig. 7.6). If an edge incident in G to such a vertex has weight h_{ij}, we assign weight $w_{ij} = 1/h_{ij}$ to the corresponding inherited edge in $\widehat{G}$.

Given an Eulerian collection T of edges in G, we construct a perfect matching in $\widehat{G}$ as follows: we include an inherited edge into the perfect matching if and only if the corresponding edge of G is not included in T. We then include auxiliary edges to complete the matching (the choice is unique). One can observe that the correspondence is a bijection between Eulerian collections in G (which are just collections of vertex-disjoint cycles) and perfect matchings in $\widehat{G}$. One can deduce from (7.4.5.3) that

$$Z_0(G) = 2^{|V|} \left(\prod_{\{i,j\} \in E} h_{ij} \right) \mathrm{haf}(\widehat{G}),$$

where $\mathrm{haf}(\widehat{G})$ is the sum of weights of the perfect matchings in $\widehat{G}$ and where the weight of a perfect matching is the product of weights of its edges, see Sect. 4.1.

We note that if G is a planar graph then $\widehat{G}$ is also planar.

Chapter 8
Partition Functions of Integer Flows

We consider yet another extension of the permanent, and some of the methods and results of Chap. 3 (capacity of polynomials, connections to $\mathbb{H}$-stable polynomials, the van der Waerden and Bregman–Minc bounds) are used. Geometrically, with each integer point of a polyhedron in $\mathbb{R}^n$, we associate a monomial in n real variables and the partition function is just the sum of monomials over the integer points in the polyhedron. When the variables are non-negative, we prove a general upper bound for the partition function in terms of the solution to a convex optimization problem (entropy maximization) on the polyhedron. Although for general polyhedra there can be no matching lower bound, such a bound indeed exists in the case of polyhedra of feasible flows in a graph. This allows us to understand what a "typical" random integer point in a flow polyhedron looks like. Based on this understanding and with intuition supplied by the Local Central Limit Theorem, we present a heuristic "Gaussian" formula for the partition function of a general polyhedron. Its validity has indeed been proven in some particular cases, though not in this book.

8.1 The Partition Function of 0-1 Flows

8.1.1 Definitions. Let us choose positive integer vectors $R = (r_1, \ldots, r_m)$ and $C = (c_1, \ldots, c_n)$ such that

$$r_1 + \ldots + r_m = c_1 + \ldots + c_n = N \tag{8.1.1.1}$$

and let $\Sigma_0(R, C)$ be the set of all $m \times n$ matrices with row sums R, column sums C and 0-1 entries:

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Algorithms and Combinatorics 30, DOI 10.1007/978-3-319-51829-9_8

$$\Sigma_0(R,C) = \Big\{ D = \big(d_{ij}\big) : \quad \sum_{j=1}^{n} d_{ij} = r_i \quad \text{for} \quad i = 1, \ldots, m$$
$$\sum_{i=1}^{n} d_{ij} = c_j \quad \text{for} \quad j = 1, \ldots, n \quad \text{and}$$
$$d_{ij} \in \{0,1\} \quad \text{for all} \quad i, j \Big\}.$$

The vectors R and C are called *margins* of a matrix from $\Sigma_0(R,C)$. The Gale–Ryser Theorem, see for example, Sect. 6.2 of [BR91], provides a convenient necessary and sufficient condition for $\Sigma_0(R,C)$ to be non-empty: assuming that

$$m \geq c_1 \geq c_2 \geq \ldots \geq c_n > 0$$

and that

$$n \geq r_i > 0 \quad \text{for} \quad i = 1, \ldots, m,$$

there is a 0-1 matrix with row sums R and column sums C if and only if the balance condition (8.1.1.1) holds and

$$\sum_{i=1}^{m} \min\{r_i, k\} \;\geq\; \sum_{j=1}^{k} c_j \quad \text{for} \quad k = 1, \ldots, n. \tag{8.1.1.2}$$

In the extreme case when $\Sigma_0(R,C)$ consists of a single matrix, that matrix has 1s arranged in a staircase pattern:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus in the above matrix we have $m = 5, n = 4$, $R = (5,3,3,2)$, $C = (4,4,3,1,1)$ and (8.1.1.2) are equalities.

Assume that $\Sigma_0(R,C)$ is indeed non-empty. Given a non-negative $m \times n$ matrix $W = \big(w_{ij}\big)$ of *weights*, we define the *partition function of 0-1 flows* by

$$\mathrm{Fl}_0(R,C;W) = \sum_{\substack{D \in \Sigma_0(R,C) \\ D = (d_{ij})}} \prod_{i,j} w_{ij}^{d_{ij}}$$

and we agree that $0^0 = 1$ so that $\mathrm{Fl}_0(R,C;W)$ remains a continuous function of W when $w_{ij} \longrightarrow 0+$.

In particular, if $m = n$ and $R = C = (1, \ldots, 1)$ then $\mathrm{Fl}_0(R,C;W) = \mathrm{per}\, W$, see Sect. 3.1.

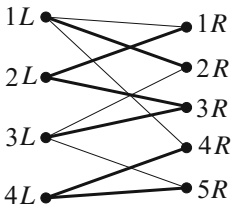

Fig. 8.1 A bipartite graph and a feasible 0-1 flow (*thick edges*) with supplies (1, 2, 1, 2) and demands (1, 1, 2, 1, 1)

8.1.2 The flow interpretation. When the matrix W of weights is itself a 0-1 matrix, the partition function $\mathrm{Fl}_0(R, C; W)$ is naturally interpreted as the number of feasible flows in the network. Namely, we consider a bipartite graph G with $m + n$ vertices numbered $1L, 2L, \ldots, mL$ and $1R, 2R, \ldots, nR$ and edges (iL, jR) whenever $w_{ij} = 1$. We assign to each vertex iL the *supply* r_i and to each vertex jR the *demand* c_j. A *feasible 0-1 flow* is a subset F of edges of G such that every vertex iL is incident to r_i edges from F and each vertex jR is incident to c_j edges from F. For example, the feasible 0-1 flow on Fig. 8.1 corresponds to $m = 4$, $n = 5$,

$$W = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and $R = (1, 2, 1, 2)$, $C = (1, 1, 2, 1, 1)$.

The following estimate of $\mathrm{Fl}_0(R, C; W)$ in terms of the capacity of a certain polynomial, see Sect. 2.4, was obtained by Gurvits [Gu15].

8.1.3 Theorem. *Given a non-negative $m \times n$ matrix $W = (w_{ij})$ of weights, we define a polynomial p_W in $m + n$ variables by*

$$p_W(x_1, \ldots, x_m; y_1, \ldots, y_n) = \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (x_i + w_{ij} y_j).$$

Given margins $R = (r_1, \ldots, r_m)$ and $C = (c_1, \ldots, c_n)$, let

$$\alpha(R, C; W) = \inf_{\substack{x_1, \ldots, x_m > 0 \\ y_1, \ldots, y_n > 0}} \frac{p_W(x_1, \ldots, x_m; y_1, \ldots, y_n)}{x_1^{n-r_1} \cdots x_m^{n-r_m} y_1^{c_1} \cdots y_n^{c_n}}.$$

Then

$$\left(\prod_{i=1}^{m} \frac{r_i^{r_i} (n - r_i)^{n-r_i} n!}{r_i! (n - r_i)! n^n} \right) \left(\prod_{j=1}^{n} \frac{c_j^{c_j} (m - c_j)^{m-c_j} m!}{c_j! (m - c_j)! m^m} \right) \alpha(R, C; W)$$
$$\le \mathrm{Fl}_0(R, C; W) \le \alpha(R, C; W).$$

Proof. First, we claim that $\mathrm{Fl}_0(R, C; W)$ is the coefficient of the monomial

$$x_1^{n-r_1} \cdots x_m^{n-r_m} y_1^{c_1} \cdots y_n^{c_n}$$

in the monomial expansion of p_W. Indeed, let us write the monomial expansion of p_W by expanding the product of mn factors $\left(x_i + w_{ij} y_j\right)$. With each monomial of p_W, we associate an $m \times n$ matrix $D = \left(d_{ij}\right)$ of 0s and 1s as follows. We let $d_{ij} = 1$ if from the factor $\left(x_i + w_{ij} y_j\right)$ we pick up the term $w_{ij} y_j$ and we let $d_{ij} = 0$ if we pick up x_i. We obtain the monomial $x_1^{n-r_1} \cdots x_m^{n-r_m} y_1^{c_1} \ldots y_n^{c_n}$ precisely when the row sums of D are $r_1, \ldots, r_m$ and the column sums of D are $c_1, \ldots, c_n$.

Next, we observe that p_W is $\mathbb{H}$-stable, see Sect. 2.4, provided $W = \left(w_{ij}\right)$ is a non-negative real matrix. Indeed, if $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ are complex variables such that $\Im\, x_1, \ldots, \Im\, x_m > 0$ and $\Im\, y_1, \ldots, \Im\, y_n > 0$ then $\Im\left(x_i + w_{ij} y_j\right) > 0$ for all i, j and hence $p_W\left(x_1, \ldots, x_m; y_1, \ldots, y_n\right) \neq 0$.

Finally, we note that the degree of x_i in p_W does not exceed n for $i = 1, \ldots, m$, while the degree of y_j in p_W does not exceed m for $j = 1, \ldots, n$. The result now follows from Theorem 2.4.7. □

Some remarks are in order. Suppose that W is a 0-1 matrix, so that $\mathrm{Fl}_0(R, C; W)$ enumerates 0-1 flows. In many asymptotic regimes, the quantity $\alpha(R, C; W)$ captures at least the logarithmic order of $\mathrm{Fl}_0(R, C; W)$. For example, if m, n, r_i and c_j grow roughly proportionately, so that $\ln \mathrm{Fl}_0(R, C; W)$ grows roughly linearly with mn, it follows from Stirling's formula

$$x! = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + O\left(x^{-1}\right)\right) \quad \text{as} \quad x \longrightarrow +\infty,$$

that $\alpha(R, C; W)$ approximates $\mathrm{Fl}_0(R, C; W)$ within a factor of $e^{O(m+n)}$, that is, $\alpha(R, C; W)$ captures the logarithmic order of $\mathrm{Fl}_0(R, C; W)$. If $m = n$ and $r_i = c_j = 1$ for all i, j then $\mathrm{Fl}_0(R, C; W) = \operatorname{per} W$ and $\alpha(R, C; W)$ approximates $\mathrm{Fl}_0(R, C; W)$ within a factor of

$$\left(1 - \frac{1}{n}\right)^{2n(n-1)} \approx e^{-2(n-1)},$$

and hence even in some sparse regimes $\alpha(R, C; W)$ captures the logarithmic order of $\mathrm{Fl}_0(R, C; W)$. If W is sparse matrix, the bounds can be improved further, see [Gu15], since the bounds in Theorem 2.4.7 can be made sharper by a more careful application of Theorem 2.4.3.

In [B10b], a weaker bound for the approximation of $\mathrm{Fl}_0(R, C; W)$ by $\alpha(R, C; W)$ was obtained. Based on that bound, it was shown that a "typical" matrix $D = \left(d_{ij}\right)$ of 0s and 1s, with row sums R and column sums C, concentrates about a particular "maximum entropy" matrix $\Theta = \left(\theta_{ij}\right)$ that maximizes the strictly concave function

$$\sum_{ij}\left(x_{ij} \ln \frac{1}{x_{ij}} + \left(1 - x_{ij}\right) \ln \frac{1}{1 - x_{ij}}\right)$$

on the polytope of $m \times n$ matrices $X = (x_{ij})$ with row sums R, column sums C and entries between 0 and 1. We discuss the connection is Sect. 8.5.1.

If $W = (1)_{m\times n}$ is the matrix filled by 1s, then $\mathrm{Fl}_0(R, C; W)$ is just the number of all 0-1 matrices with row sums R and column sums C. There is an extensive literature on approximate and asymptotic formulas for $\mathrm{Fl}_0(R, C; W)$, see [G+06, C+08, BH13, IM16] and references therein.

Since $\mathrm{Fl}_0(R, C; W)$ can be represented as the coefficient of a monomial in a product of mn linear forms, it follows from Sect. 3.2.1 that $\mathrm{Fl}_0(R, C; W)$ can be represented as the permanent of an $(mn)\times(mn)$ matrix. For example, for $m = n = 3$, $R = (3, 2, 1)$ and $C = (2, 2, 2)$, formula (3.2.1.2) gives

$$\mathrm{Fl}_0(R, C; W) = \frac{1}{0!1!2!2!2!2!}\mathrm{per}\begin{pmatrix}
0&0&0&w_{11}&w_{11}&0&0&0&0\\
0&0&0&0&0&w_{12}&w_{12}&0&0\\
0&0&0&0&0&0&0&w_{13}&w_{13}\\
1&0&0&w_{21}&w_{21}&0&0&0&0\\
1&0&0&0&0&w_{22}&w_{22}&0&0\\
1&0&0&0&0&0&0&w_{23}&w_{23}\\
0&1&1&w_{31}&w_{31}&0&0&0&0\\
0&1&1&0&0&w_{32}&w_{32}&0&0\\
0&1&1&0&0&0&0&w_{33}&w_{33}
\end{pmatrix}$$

which simplifies to

$$\begin{aligned}\mathrm{Fl}_0(R, C; W) =& w_{11}w_{31}w_{12}w_{22}w_{13}w_{23} + w_{11}w_{21}w_{12}w_{32}w_{13}w_{23}\\ &+ w_{11}w_{21}w_{12}w_{22}w_{13}w_{33}.\end{aligned}$$

As Jerrum, Sinclair and Vigoda remark in [J+04], their randomized polynomial time algorithm for approximating the permanent of a non-negative real matrix can be applied to approximate $\mathrm{Fl}_0(R, C; W)$ in polynomial time.

8.2 The Partition Function of Integer Flows

8.2.1 Definitions. Let us choose positive integer vectors $R = (r_1, \ldots, r_m)$ and $C = (c_1, \ldots, c_n)$ such that

$$r_1 + \ldots + r_m = c_1 + \ldots + c_n = N \tag{8.2.1.1}$$

and let $\Sigma_+(R, C)$ be the set of all $m \times n$ non-negative integer matrices with row sums R and column sums C:

$$\Sigma_+(R,C) = \Big\{ D = \big(d_{ij}\big) : \quad \sum_{j=1}^{n} d_{ij} = r_i \quad \text{for} \quad i = 1, \ldots, m,$$
$$\sum_{i=1}^{n} d_{ij} = c_j \quad \text{for} \quad j = 1, \ldots, n,$$
$$d_{ij} \in \mathbb{Z} \quad \text{and} \quad d_{ij} \geq 0 \quad \text{for all} \quad i, j \Big\}.$$

The vectors R and C are called *margins* of a matrix from $\Sigma_+(R,C)$. It is not hard to show that for positive integer vectors R and C the set $\Sigma_+(R,C)$ is non-empty if and only if the balance condition (8.2.1.1) is satisfied. Assuming that $\Sigma_+(R,C)$ is non-empty, for an $m \times n$ non-negative matrix $W = \big(w_{ij}\big)$ of *weights*, we define the *partition function of integer flows* by

$$\mathrm{Fl}_+(R,C;W) = \sum_{\substack{D \in \Sigma_+(R,C) \\ D = (d_{ij})}} \prod_{ij} w_{ij}^{d_{ij}}.$$

As in Sect. 8.1, we agree that $0^0 = 1$, so that $\mathrm{Fl}_+(R,C;W)$ remains a continuous function of W when $w_{ij} \longrightarrow 0+$.

8.2.2 The flow interpretation. Suppose that $w_{ij} \in \{0,1\}$ for all i, j. Then $\mathrm{Fl}_+(R,C;W)$ is interpreted as the number of integer feasible flows in a network. As in Sect. 8.1.2, we consider a bipartite graph G with $m+n$ vertices numbered $1L, \ldots, mL$ and $1R, \ldots, nR$ and edges (iL, jR) whenever $w_{ij} = 1$. We assign to each vertex iL the *supply* r_i and to each vertex jR the *demand* c_j. A *feasible integer flow* is an assignment of non-negative integer numbers to the edges of G so that for every vertex iL the sum of the assigned numbers on the incident edges is r_i while for every vertex jR the sum of the assigned numbers on the incident edges is c_j.

More generally, suppose that G is a directed graph without loops or multiple edges. Suppose that to every vertex v of G an integer $a(v)$ is assigned, which can be positive ("demand"), negative ("supply") or 0 ("transit"). A *feasible integer flow* is an assignment of non-negative integers $x(e)$ to every edge e of G so that for every v the balance condition inflow $-$ outflow $= a(v)$ is satisfied, see Fig. 8.2:

$$\sum_{e : e = u \to v} x(e) - \sum_{e : e = v \to u} x(e) = a(v) \quad \text{for all} \quad v.$$

Given that the set of feasible integer flows is non-empty, it is finite, if and only if the graph contains no directed cycles of the type $v_1 \to v_2 \to \ldots \to v_n \to v_1$. If there are no directed cycles, one can construct a bipartite graph $\widehat{G}$ and a bijection between the set of feasible integer flows in G and the set of feasible integer flows in $\widehat{G}$ as follows. With every vertex v of G we associate two vertices vL and vR of $\widehat{G}$, connected by a directed edge $vL \to vR$. For every edge $v \to u$ of G, we introduce the edge $vL \to uR$ of $\widehat{G}$. For every vertex v of G, we choose a positive integer $z(v)$

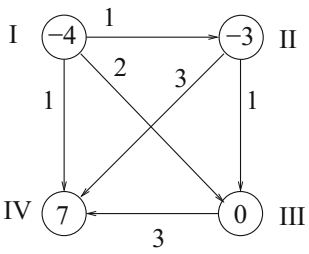

Fig. 8.2 A graph with 4 vertices and a feasible flow, corresponding to demands/supplies written inside each vertex

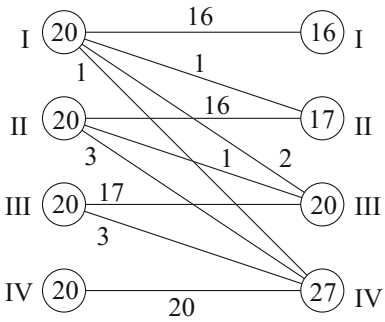

Fig. 8.3 The feasible integer flow in a bipartite graph with $z(v) = 20$ for all v, corresponding to the feasible integer flow on Fig. 8.2

which is at least as large as a possible outflow from v. We let the supply in vL equal to $z(v)$ and demand in vR equal to $z(v) + a(v)$, where $a(v)$ is the demand/supply in v. To construct the bijection, if $x(e)$ for $e = v \to u$ is a flow in G, we introduce the flow $x(e)$ on the edge $vL \to uR$ in $\widehat{G}$ and for every vertex v of G, we introduce the flow of $z(v)-$ outflow of v on the edge $vL \longrightarrow vR$, see Fig. 8.3.

Hence the number of integer feasible flows in an arbitrary directed graph without directed cycles can be encoded as $\mathrm{Fl}_+(R, C; W)$ for appropriate R, C and W.

In fact, we can also incorporate upper bounds on the size of flow on edges, that is, enumerate integer feasible flows $x(e)$ in G with additional constraints $x(e) \le c(e)$, where $c(e)$ are given positive integers (frequently referred to as *capacities of edges*). For that, for every edge $v \to u$, we introduce two auxiliary vertices w_+ and w_-, replace the edge $v \to u$ by the three edges $v \to w_+$, $w_- \to w_+$ and $w_- \to u$, and let $a(w_+) = c(v \to u)$ and $a(w_-) = -c(v \to u)$. Then a flow on $x(e)$ on the edge $v \to u$ in G satisfying $x(e) \le c(e)$, corresponds to the flow $x(e)$ on the edge $v \to w_+$, flow $c(e) - x(e)$ on the edge $w_- \to w_+$ and flow $x(e)$ on the edge $w_- \to u$, see Fig. 8.4.

In particular, we can express the number $\mathrm{Fl}_0(R, C; W)$ of feasible 0-1 flows, see Sect. 8.1, as the number $\mathrm{Fl}_+(R', C', W')$ of feasible integer flows. One particular case resulting in the *Kostant partition function of type A* is of interest to representation theory, see [BV09]. There we are interested in the number of feasible integer flows in a graph with vertices numbered $1, \ldots, n$ and edges $i \to j$ for $j > i$, see also Fig. 8.2 for an example.

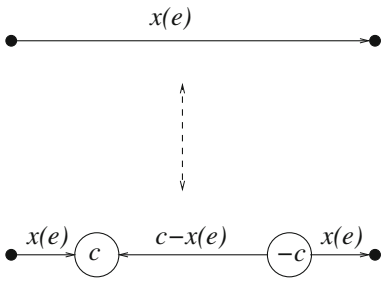

Fig. 8.4 Enforcing the condition $x(e) \le c$

We will use a representation of $\mathrm{Fl}_+(R, C; W)$ as the permanent of a structured random matrix. Recall that a random variable ω is *standard exponential* if the density of ω is

$$\begin{cases} e^{-t} & \text{if } t > 0 \\ 0 & \text{if } t \le 0. \end{cases}$$

Recall that

$$\mathbf{E}\,\omega^k = \int_0^{+\infty} t^k e^{-t}\,dt = k!$$

for non-negative integer k.

The following result was obtained in [Ba07].

8.2.3 Lemma. *Let $\Omega = \big(\omega_{ij}\big)$ be the $m \times n$ matrix of independent standard exponential random variables ω_{ij}. Given positive integer vectors $R = (r_1, \ldots, r_m)$ and $C = (c_1, \ldots, c_n)$ such that*

$$r_1 + \ldots + r_m = c_1 + \ldots + c_n = N$$

and an $m \times n$ matrix $W = \big(w_{ij}\big)$ of weights, let us construct a random $N \times N$ matrix $A(\Omega) = A_{R,C;W}(\Omega)$ as follows: the rows of $A(\Omega)$ are split into m blocks, with the i-th block containing r_i rows, the columns of A are split into n blocks, with the jth block containing c_j columns and the entries in the (i, j)-th block are all equal to $w_{ij}\omega_{ij}$, see Fig. 8.5.

Then

$$\mathrm{Fl}_+(R, C; W) = \frac{\mathbf{E}\,\mathrm{per}\,A(\Omega)}{r_1! \cdots r_m! c_1! \cdots c_n!}.$$

Proof. Let us pick one entry from every row and every column of $A(\Omega)$ and let d_{ij} be the number of entries picked from the (i, j)-th block. Clearly, $D = \big(d_{ij}\big)$ is an $m \times n$ non-negative integer matrix with row sums R and column sums C and the expectation of the product of the picked entries is

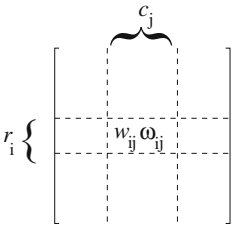

Fig. 8.5 The structure of the matrix $A(\Omega)$

$$\mathbf{E}\prod_{i,j}\left(w_{ij}\omega_{ij}\right)^{d_{ij}}=\prod_{i,j}d_{ij}!w_{ij}^{d_{ij}}. \tag{8.2.3.1}$$

Let us now compute how many times we obtain a given non-negative integer matrix $D=\left(d_{ij}\right)$ with row sums R and column sums C. For that, for $i=1,\ldots,m$, we split the i-th block of r_i rows into n sub-blocks with $d_{i1},\ldots,d_{in}$ rows in

$$\prod_{i=1}^{m}\frac{r_i!}{d_{i1}!\cdots d_{in}!}$$

ways, for $j=1,\ldots,n$, we split the j-th block of c_j columns into m sub-blocks with $d_{1j},\ldots,d_{mj}$ columns in

$$\prod_{j=1}^{n}\frac{c_j!}{d_{1j}!\cdots d_{mj}!}$$

ways and then for each i and j such that $d_{ij}>0$ we choose one entry in every row of the j-th sub-block of the i-th block of rows and every column of the i-th sub-block of the j-th block of columns altogether in

$$\prod_{i,j}^{n}d_{ij}!$$

ways, see Fig. 8.6.

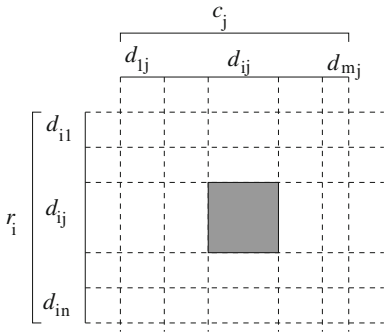

Fig. 8.6 Subdividing rows and columns further into sub-blocks

Hence we obtain (8.2.3.1) in

$$\left(\prod_{i=1}^{m} r_i!\right)\left(\prod_{j=1}^{n} c_j!\right)\left(\prod_{ij} d_{ij}!\right)^{-1}$$

ways total and

$$\mathbf{E}\,\mathrm{per}\,A(\Omega) = \left(\prod_{i=1}^{m} r_i!\right)\left(\prod_{j=1}^{n} c_j!\right)\mathrm{Fl}_+(R, C; W),$$

which completes the proof. □

8.3 Approximate Log-Concavity

For a non-negative integer vector $R = (r_1, \ldots, r_m)$, we denote

$$|R| = \sum_{i=1}^{m} r_i \quad \text{and} \quad \gamma(R) = \prod_{i=1}^{m} \frac{r_i!}{r_i^{r_i}}.$$

Our goal is to prove the following result from [Ba07].

8.3.1 Theorem. *Let W be an $m \times n$ non-negative real matrix, let $R_1, \ldots, R_k$ be non-negative integer m-vectors and let $C_1, \ldots, C_k$ be non-negative integer n-vectors such that $|R_i| = |C_i| = N$ for all i. Suppose further that $\alpha_1, \ldots, \alpha_k \geq 0$ are reals such that $\alpha_1 + \ldots + \alpha_k = 1$ and such that*

$$R = \sum_{i=1}^{k} \alpha_i R_i \quad \textit{and} \quad C = \sum_{i=1}^{k} \alpha_i C_i$$

are positive integer vectors.

Then

$$\frac{N^N}{N!}\gamma(R)\gamma(C)\mathrm{Fl}_+(R, C; W) \;\geq\; \prod_{i=1}^{k}\Big(\mathrm{Fl}_+(R_i, C_i; W)\max\{\gamma(R_i), \gamma(C_i)\}\Big)^{\alpha_i}.$$

Theorem 8.3.1 implies an approximate log-concavity of the numbers $\mathrm{Fl}_+(R, C; W)$.

8.3.2 Corollary. *Let W be an $m \times n$ non-negative real matrix, let $R_1, \ldots, R_k$ be non-negative integer m-vectors and let $C_1, \ldots, C_k$ be non-negative integer n-vectors*

such that $|R_i| = |C_i| = N$ *for all* i. *Suppose further that* $\alpha_1, \ldots, \alpha_k \geq 0$ *are reals such that* $\alpha_1 + \ldots + \alpha_k = 1$ *and such that*

$$R = \sum_{i=1}^{k} \alpha_i R_i \quad \textit{and} \quad C = \sum_{i=1}^{k} \alpha_i C_i$$

are positive integer vectors. Assuming that $R = (r_1, \ldots, r_m)$ *and* $C = (c_1, \ldots, c_n)$, *we have*

$$\frac{N^N}{N!} \min \left\{ \prod_{i=1}^{m} \frac{r_i!}{r_i^{r_i}}, \prod_{j=1}^{n} \frac{c_j!}{c_j^{c_j}} \right\} \mathrm{Fl}_+(R, C; W) \;\geq\; \prod_{i=1}^{k} \left(Fl_+ (R_i, C_i; W)\right)^{\alpha_i} .$$

From Stirling's formula,

$$\begin{aligned} \frac{N^N}{N!} =& \frac{e^N}{\sqrt{2\pi N}} \left(1 + O(1/N)\right), \quad \frac{r_i!}{r_i^{r_i}} = e^{-r_i} \sqrt{2\pi r_i} \left(1 + O(1/r_i)\right) \\ \text{and} \quad \frac{c_j!}{c_j^{c_j}} =& e^{-c_j} \sqrt{2\pi c_j} \left(1 + O(1/c_j)\right), \end{aligned}$$

it follows that

$$\frac{N^N}{N!} \min \left\{ \prod_{i=1}^{m} \frac{r_i!}{r_i^{r_i}}, \prod_{j=1}^{n} \frac{c_j!}{c_j^{c_j}} \right\} = \min \left\{ 2^{O(m)} \sqrt{r_1 \cdots r_m},\; 2^{O(n)} \sqrt{c_1 \cdots c_n} \right\}.$$

There seem to be neither a counter-example nor a proof of a hypothetical stronger inequality, which claims genuine log-concavity of $\mathrm{Fl}_+(R, C; W)$:

$$\mathrm{Fl}_+(R, C; W) \stackrel{?}{\geq} \prod_{i=1}^{k} \Big(\mathrm{Fl}_+ (R_i, C_i, ; W) \Big)^{\alpha_i}.$$

The proof of Theorem 8.3.1 uses the permanental representation of $\mathrm{Fl}_+(R, C; W)$ of Lemma 8.2.3, matrix scaling (see Sect. 3.5), the van der Waerden (Sect. 3.3) and Bregman–Minc (Sect. 3.4) inequalities.

For an $m \times n$ positive real matrix $B = \left(b_{ij}\right)$, we define a function $g_B : \mathbb{R}^m \oplus \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$g_B(x, y) = \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} b_{ij} e^{\xi_i + \eta_j} \quad \text{for} \quad x = (\xi_1, \ldots, \xi_m) \quad \text{and} \quad y = (\eta_1, \ldots, \eta_n).$$

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product in Euclidean space. For $R \in \mathbb{R}^m$ and $C \in \mathbb{R}^n$, we define a subspace $\mathcal{L}_{R,C} \subset \mathbb{R}^m \oplus \mathbb{R}^n$ by

$$\mathcal{L}_{R,C} = \left\{(x, y) \in \mathbb{R}^m \oplus \mathbb{R}^n : \quad \langle R, x\rangle = \langle C, y\rangle = 0\right\}.$$

We further define a function $f_{R,C}$ of B by

$$f_{R,C}(B) = \inf_{(x,y)\in\mathcal{L}_{R,C}} g_B(x, y) \ \geq \ 0,$$

see Theorem 3.5.8.

8.3.3 Lemma. *Let $R = (r_1, \ldots, r_m)$ be a positive integer m-vector and let $C = (c_1, \ldots, c_n)$ be a positive integer n-vector such that $|R| = |C| = N$. Let $W = \left(w_{ij}\right)$ be an $m \times n$ positive matrix. For an $m \times n$ matrix $\Omega = \left(\omega_{ij}\right)$ of independent standard exponential random variables, let us define the $m \times n$ matrix $B = B(\Omega) = \left(b_{ij}\right)$ by $b_{ij} = w_{ij}\omega_{ij}$ for all i, j. Then*

$$\frac{N!}{N^N}\left(\prod_{i=1}^m \frac{r_i^{r_i}}{r_i!}\right)\left(\prod_{j=1}^n \frac{c_j^{c_j}}{c_j!}\right)\frac{1}{N^N}\mathbf{E}\, f_{R,C}^N(B) \ \leq \ \mathrm{Fl}_+(R, C; W)$$
$$\leq \ \min\left\{\prod_{i=1}^m \frac{r_i^{r_i}}{r_i!}, \ \prod_{j=1}^n \frac{c_j^{c_j}}{c_j!}\right\}\frac{1}{N^N}\mathbf{E}\, f_{R,C}^N(B).$$

Proof. With probability 1, the matrix B is positive. Using Theorem 3.5.8, we scale B to a matrix with row sums $r_1, \ldots, r_m$ and column sums $c_1, \ldots, c_n$. That is, we compute a positive $m \times n$ matrix $L = L(\Omega)$, $L = \left(l_{ij}\right)$ with row sums R and column sums C and positive $\lambda_i = \lambda_i(\Omega)$ for $i = 1, \ldots, m$ and $\mu_j = \mu_j(\Omega)$ for $j = 1, \ldots, n$, such that

$$b_{ij} = l_{ij}\lambda_i\mu_j \quad \text{for all} \quad i, j.$$

By Theorem 3.5.8, we can choose

$$\lambda_i = e^{-\xi_i}\sqrt{\frac{f_{R,C}(B)}{N}} \quad \text{and} \quad \mu_j = e^{-\eta_j}\sqrt{\frac{f_{R,C}(B)}{N}},$$

where $x^* = (\xi_1, \ldots, \xi_m)$ and $y^* = (\mu_1, \ldots, \mu_n)$ is the minimum point of $g_{R,C}(B)$ on $\mathcal{L}_{R,C}$. It follows that

$$\left(\prod_{i=1}^m \lambda_i^{r_i}\right)\left(\prod_{j=1}^n \mu_j^{c_j}\right) = \frac{f_{R,C}^N(B)}{N^N}\exp\left\{-\sum_{i=1}^m r_i\xi_i\right\}\exp\left\{-\sum_{j=1}^n c_j\eta_j\right\}$$
$$= \frac{f_{R,C}^N(B)}{N^N}. \tag{8.3.3.1}$$

Let $A(\Omega)$ be the $N \times N$ matrix constructed in Lemma 8.2.3. Let us divide the entries in the (i, j)-th block of $A(\Omega)$ by $\lambda_i\mu_j r_i c_j$ and let $D = D(\Omega)$ be the $N \times N$ matrix

we obtain. Then

$$\begin{aligned}\operatorname{per} A &= \left(\prod_{i=1}^{m}(r_i\lambda_i)^{r_i}\right)\left(\prod_{j=1}^{n}(c_j\mu_j)^{c_j}\right)\operatorname{per} D \\ &= \left(\prod_{i=1}^{m} r_i^{r_i}\right)\left(\prod_{j=1}^{n} c_j^{c_j}\right)\frac{f_{r,c}^N(B)}{N^N}\operatorname{per} D \qquad (8.3.3.2)\end{aligned}$$

by (8.3.3.1). It is not hard to see that the matrix D is doubly stochastic, and hence by the van der Waerden bound, see Theorem 3.3.2, we have

$$\operatorname{per} D \;\geq\; \frac{N!}{N^N}.$$

On the other hand, the entries of the (i, j)-th block of D can be written as $l_{ij}/r_i c_j$ and hence do not exceed $\min\{1/r_i, 1/c_j\}$. Therefore, by the Bregman–Minc bound, see Corollary 3.4.5, we have

$$\operatorname{per} D \;\leq\; \min\left\{\prod_{i=1}^{m}\frac{r_i!}{r_i^{r_i}},\ \prod_{j=1}^{n}\frac{c_j!}{c_j^{c_j}}.\right\}$$

Hence from (8.3.3.2),

$$\begin{aligned}&\min\left\{\prod_{i=1}^{n} r_i!\prod_{j=1}^{n} c_j^{c_j},\ \prod_{j=1}^{n} c_j!\prod_{i=1}^{m} r_i^{r_i}\right\}\frac{f_{R,C}^N(B)}{N^N} \;\geq\; \operatorname{per} A \\ &\geq \left(\prod_{i=1}^{m} r_i^{r_i}\right)\left(\prod_{j=1}^{n} c_j^{c_j}\right)\frac{N!}{N^N}\frac{f_{R,C}^N(B)}{N^N}.\end{aligned}$$

The proof now follows from Lemma 8.2.3. □

Next, we establish some convex properties of $f_{R,C}(B)$. We define $u_d \in \mathbb{R}^d$ by $u_d = (1, \ldots, 1)$ (the d-vector of all 1s) and note that

$$g_B\big(x + \alpha u_m,\ y + \beta u_n\big) = e^{\alpha+\beta} g_B(x, y) \quad \text{for all} \quad \alpha, \beta \in \mathbb{R}.$$

8.3.4 Lemma. *Let $R_1, \ldots, R_k$ be m-vectors and let $C_1, \ldots, C_k$ be n-vectors such that*

$$\langle R_i, u_m\rangle = \langle C_i, u_n\rangle = 1 \quad \textit{for} \quad i = 1, \ldots, k.$$

Let $B_1, \ldots, B_k$ be positive real $m \times n$ matrices and let $\alpha_1, \ldots, \alpha_k \geq 0$ be real such that $\alpha_1 + \ldots + \alpha_k = 1$. Let

$$R = \sum_{i=1}^{k} \alpha_i R_i, \quad C = \sum_{i=1}^{k} \alpha_i C_i \quad \textit{and} \quad B = \sum_{i=1}^{k} \alpha_i B_i.$$

Then

$$f_{R,C}(B) \;\geq\; \prod_{i=1}^{k} \left(f_{R_i,C_i}(B_i)\right)^{\alpha_i}.$$

Proof. Let us choose a point $(x, y) \in \mathcal{L}_{R,C}$, so that

$$\langle R, x\rangle = \langle C, y\rangle = 0. \tag{8.3.4.1}$$

We define

$$x_i = x - \langle R_i, x\rangle u_m \quad \text{and} \quad y_i = y - \langle C_i, y\rangle u_n \quad \text{for} \quad i = 1, \ldots, k.$$

Hence

$$\begin{aligned} \langle R_i, x_i\rangle &= \langle R_i, x\rangle - \langle R_i, x\rangle\langle R_i, u_m\rangle = 0 \quad \text{and} \\ \langle C_i, y_i\rangle &= \langle C_i, y\rangle - \langle C_i, y\rangle\langle C_i, u_n\rangle = 0, \end{aligned}$$

so that

$$(x_i, y_i) \in \mathcal{L}_{R_i,C_i}. \tag{8.3.4.2}$$

Then

$$g_B(x, y) = \sum_{i=1}^{k} \alpha_i g_{B_i}(x, y) \;\geq\; \prod_{i=1}^{k} \left(g_{B_i}(x, y)\right)^{\alpha_i} \tag{8.3.4.3}$$

and

$$g_{B_i}(x, y) = g_{B_i}\left(x_i + \langle R_i, x\rangle u_m,\; y_i + \langle C_i, y\rangle u_n\right) = e^{\langle R_i, x\rangle} e^{\langle C_i, y\rangle} g_{B_i}(x_i, y_i). \tag{8.3.4.4}$$

Since by (8.3.4.1)

$$\begin{aligned} \prod_{i=1}^{k} \left(e^{\langle R_i, x\rangle}\right)^{\alpha_i} &= \exp\left\{\left\langle \sum_{i=1}^{k} \alpha_i R_i,\; x \right\rangle\right\} = \exp\{\langle R, x\rangle\} = 1 \quad \text{and} \\ \prod_{i=1}^{k} \left(e^{\langle C_i, y\rangle}\right)^{\alpha_i} &= \exp\left\{\left\langle \sum_{i=1}^{k} \alpha_i C_i,\; y \right\rangle\right\} = \exp\{\langle C, y\rangle\} = 1, \end{aligned}$$

combining (8.3.4.2)–(8.3.4.4) we obtain

$$g_B(x, y) \ \geq \ \prod_{i=1}^{k} \left(g_{B_i}(x_i, y_i)\right)^{\alpha_i} \ \geq \ \prod_{i=1}^{k} \left(f_{R_i,C_i}(B_i)\right)^{\alpha_i}.$$

Sine the point $(x, y) \in \mathcal{L}_{R,C}$ was chosen arbitrarily, the proof follows. □

8.3.5 Proof of Theorem 8.3.1. By continuity, it suffices to prove Theorem 8.3.1 assuming, additionally, that matrix $W = \left(w_{ij}\right)$ is positive.

Given an $m \times n$ matrix Ω of independent standard exponential random variables, let us construct a random matrix $B = B(\Omega)$ as in Lemma 8.3.3. By Lemma 8.3.3, we have

$$\frac{N^N}{N!}\gamma(R)\gamma(C)\mathrm{Fl}_+(R, C; W) \ \geq \ \frac{1}{N^N}\mathbf{E}\, f_{R,C}^N(B) \quad \text{and}$$
$$\max\left\{\gamma(R_i), \gamma(C_i)\right\}\mathrm{Fl}_+(R_i, C_i; W) \ \leq \ \frac{1}{N^N}\mathbf{E}\, f_{R_i,C_i}^N(B) \quad \text{for} \quad i = 1, \ldots, k.$$

If $\Omega_1, \ldots, \Omega_k$ are different realizations of Ω and

$$\Omega_0 = \sum_{i=1}^{k} \alpha_i \Omega_i,$$

then for the corresponding matrices $B_i = B(\Omega_i)$ for $i = 0, 1, \ldots, k$, we have

$$B_0 = \sum_{i=1}^{k} \alpha_i B_i$$

and by Lemma 8.3.4,

$$f_{R,C}^N(B_0) \ \geq \ \prod_{i=1}^{k} \left(f_{R_i,C_i}^N(B_i)\right)^{\alpha_i}.$$

Note that we can apply Lemma 8.3.4 since $|R_i| = |C_i| = N$ for all i, so that

$$f_{R_i,C_i} = f_{R_i/N,C_i/N} \quad \text{and} \quad f_{R,C} = f_{R/N,C/N}$$

and the sum of the coordinates of vectors R_i/N, C_i/N, R/N and C/N are equal to 1.

Since the density of the random matrix $\Omega = \left(\omega_{ij}\right)$ is

$$\begin{cases} \prod_{ij} e^{-t_{ij}} & \text{if } t_{ij} > 0 \text{ for all } i, j \\ 0 & \text{otherwise,} \end{cases}$$

applying the Prékopa–Leindler inequality of Sect. 2.1.6, we conclude that

$$\mathbf{E}\, f^N_{R,C}(B) \;\geq\; \prod_{i=1}^{k} \left(\mathbf{E}\, f^N_{R_i,C_i}(B)\right)^{\alpha_i},$$

which completes the proof. □

8.3.6 Proof of Corollary 8.3.2. Using that the function

$$x \longmapsto \ln \frac{\Gamma(x+1)}{x^x} \quad \text{for} \quad x \geq 1$$

is concave, we conclude that

$$\gamma(R) \;\geq\; \prod_{i=1}^{k} \gamma^{\alpha_i}(R_i) \quad \text{and} \quad \gamma(C) \;\geq\; \prod_{i=1}^{k} \gamma^{\alpha_i}(C_i)$$

and the proof follows from Theorem 8.3.1. □

8.4 Bounds for the Partition Function

Corollary 8.3.2 allows us to estimate the partition function $\mathrm{Fl}_+(R, C; W)$ in terms of the capacity of a certain polynomial.

8.4.1 Complete symmetric polynomial. The complete symmetric polynomial $h_N(z_1, \ldots, z_d)$ of degree N in d variables $z_1, \ldots, z_d$ is the sum of all $\binom{N+d-1}{d-1}$ monomials in $z_1, \ldots, z_d$ of the total degree N. It can be defined recursively as $h_N(z_1) = z_1^N$ and

$$h_N(z_1, \ldots, z_d) = \sum_{m=0}^{N} z_d^m\, h_{N-d}(z_1, \ldots, z_{d-1}),$$

which also provides a fast way to compute h_N at any given $z_1, \ldots, z_d$.

8.4.2 Theorem. *Let $R = (r_1, \ldots, r_m)$ be a positive integer m-vector and let $C = (c_1, \ldots, c_n)$ be a positive integer n-vector such that*

$$r_1 + \ldots + r_m = c_1 + \ldots + c_n = N.$$

Let $W = \left(w_{ij}\right)$ be a non-negative real $m \times n$ matrix of weights. Let us define a polynomial $p = p_{R,C;W}$ in $m + n$ real variables $x_1, \ldots, x_m; y_1, \ldots, y_n$ by

$$p(x_1, \ldots, x_m; y_1, \ldots, y_n) = h_N(z_{ij}) \quad \textit{for} \quad z_{ij} = w_{ij} x_i y_j,$$

where $h_N(z_{ij})$ is the complete symmetric polynomial of degree N in mn variables z_{ij} where $1 \leq i \leq m$ and $1 \leq j \leq n$. Let

$$\alpha_+(R, C; W) = \inf_{\substack{x_1,\ldots,x_m>0\\ y_1,\ldots,y_n>0}} \frac{p\,(x_1, \ldots, x_m; y_1, \ldots, y_n)}{x_1^{r_1} \cdots x_m^{r_m} y_1^{c_1} \cdots y_n^{c_n}}.$$

Then

$$\binom{N+m-1}{m-1}^{-1} \binom{N+n-1}{n-1}^{-1} \frac{N!}{N^N} \max\left\{\prod_{i=1}^{m} \frac{r_i^{r_i}}{r_i!}, \prod_{j=1}^{n} \frac{c_j^{c_j}}{c_j!}\right\} \alpha_+(R, C)$$
$$\leq \mathrm{Fl}_+(R, C; W) \leq \alpha_+(R, C; W).$$

Proof. Expanding p into the sum of monomials, we get

$$p\,(x_1, \ldots, x_m; y_1, \ldots, y_n) = \sum_{\substack{A=(a_1,\ldots,a_m)\\ B=(b_1,\ldots,b_n)}} \mathrm{Fl}_+(A, B; W) x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_n^{b_n}$$

where the sum is taken over all pairs non-negative integer m-vectors $(a_1, \ldots, a_m)$ and n-vectors $(b_1, \ldots, b_n)$ such that $a_1 + \ldots + a_m = b_1 + \ldots + b_n = N$. Hence the upper bound is immediate. As the total number of monomials is $\binom{N+m-1}{m-1}\binom{N+n-1}{n-1}$, the lower bound follows from Corollary 8.3.2 and (2.1.5.3). □

Theorem 8.4.2 shows that $\alpha_+(R, C; W)$ approximates $\mathrm{Fl}_+(R, C; W)$ within a factor of $N^{O(m+n)}$ (the implicit constant in the "O" notation is absolute). In many interesting asymptotic regimes, $\alpha_+(R, C; W)$ captures the logarithmic asymptotics of $\mathrm{Fl}_+(R, C; W)$. A similar, though less explicit, bound

$$\frac{\beta_+(R, C; W)}{N^{O(m+n)}} \leq \mathrm{Fl}_+(R, C; W) \leq \beta_+(R, C; W)$$

where

$$\beta_+(R, C; W) = \inf_{\substack{x_1,\ldots,x_m>0\\ y_1,\ldots,y_n>0\\ w_{ij}x_iy_j<1 \text{ for all } i,j}} \left(\prod_{i=1}^{m} x_i^{-r_i}\right)\left(\prod_{j=1}^{n} y_j^{-c_j}\right) \prod_{i,j} \frac{1}{1 - w_{ij}x_iy_j}$$

was obtained in [Ba09]. Based on it, it was shown in [B10a] that a "typical" random non-negative integer matrix $D = (d_{ij})$ with row sums R and column sums C concentrates about a particular "maximum entropy" matrix matrix $\Theta = (\theta_{ij})$ that maximizes the strictly concave function

$$\sum_{ij}\Big((x_{ij} + 1)\ln(x_{ij} + 1) - x_{ij}\ln x_{ij}\Big)$$

on the polytope of non-negative real matrices $X = \left(x_{ij}\right)$ with row sums R and column sums C. We discuss the connection in Sect. 8.5.2.

If $W = (1)_{m \times n}$ is the matrix filled with 1s, then $\mathrm{Fl}_+(R, C; W)$ is just the number of non-negative integer matrices with row sums R and column sums C. There is an extensive literature on asymptotic and approximate formulas for the number of such matrices, see [DG95, GM08, CM10, BH12, IM16] and references therein.

8.5 Concluding Remarks: Partition Functions for Integer Points in Polyhedra

The partition functions $\mathrm{Fl}_0(R, C; W)$ and $\mathrm{Fl}_+(R, C; W)$ can be considered as special cases of more general partition functions for 0-1, respectively integer, points in polyhedra.

8.5.1 Partition function of 0-1 points. Let $A = \left(a_{ij}\right)$ be an integer $r \times n$ matrix of rank r, let $b = (b_1, \ldots, b_r)$ be an integer r-vector, and let $w = (w_1, \ldots, w_n)$ be a positive real vector of weights. We consider the set $\mathcal{X}_0(A, B)$ of the 0-1 vectors x that lie in the affine subspace defined by the system $Ax = b$:

$$\mathcal{X}_0(A, b) = \Big\{ x = (x_1, \ldots, x_n) : \sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{for} \quad i = 1, \ldots, r \quad \text{and} \quad x_j \in \{0, 1\} \quad \text{for} \quad j = 1, \ldots, n \Big\}.$$

We define a weighted sum (partition function) over $\mathcal{X}_0(A, b)$

$$S_0(A, b; w) = \sum_{\substack{x \in \mathcal{X}_0(A,b) \\ x=(x_1,\ldots,x_n)}} w_1^{x_1} \cdots w_n^{x_n}.$$

It is not very hard to come up with an upper bound similar to the bound α_0 of Sect. 8.1:

$$S_0(A, b; w) \;\le\; \inf_{t_1, \ldots, t_r > 0} t_1^{-b_1} \cdots t_r^{-b_r} \prod_{j=1}^{n} \left(1 + w_j t_1^{a_{1j}} \cdots t_r^{a_{rj}}\right). \tag{8.5.1.1}$$

In general, there is no non-trivial lower bound for $S_0(A, b; w)$ since there is no guarantee that the set $\mathcal{X}_0(A, b)$ of 0-1 vectors satisfying a given system of linear equations is non-empty. There is, however, a dual reformulation of (8.5.1.1) which leads to sharper upper bounds and, sometimes, to good approximations of $S_0(A, b; w)$.

Let $P_0(A, b)$ be the polyhedron that is the intersection of the cube $[0, 1]^n$ with the affine subspace defined by the system $Ax = b$,

$$P_0(A,b) = \Big\{ x = (x_1, \ldots, x_n) : \sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{for} \quad i = 1, \ldots, r \quad \text{and} \quad 0 \le x_j \le 1 \quad \text{for} \quad j = 1, \ldots, n \Big\}.$$

Suppose that $P_0(A,b)$ has a non-empty relative interior, that is, contains a point $x = (x_1, \ldots, x_n)$ such that $0 < x_j < 1$ for $j = 1, \ldots, n$. Let us consider a strictly concave function

$$H_w\left(x_1, \ldots, x_j\right) = \sum_{j=1}^{n} \left(x_j \ln \frac{1}{x_j} + (1 - x_j) \ln \frac{1}{1 - x_j} + x_j \ln w_j \right)$$
$$\text{where} \quad 0 \le x_j \le 1 \quad \text{for} \quad j = 1, \ldots, n.$$

It is not hard to show that H_w attains its maximum on $P_0(a,b)$ at a unique point $(\xi_1, \ldots, \xi_n)$ in the relative interior of P, cf. the proof of Theorem 3.5.2 and see [BH10] for detail, and we claim that

$$S_0(A, b; w) \;\le\; \exp\left\{H_w\left(\xi_1, \ldots, \xi_n\right)\right\} \tag{8.5.1.2}$$

and, moreover, that the bounds of (8.5.1.1) and (8.5.1.2) are identical.

Indeed, the Lagrange multiplier optimality condition implies that for some real $\lambda_1, \ldots, \lambda_r$, we have

$$\ln\left(\frac{1 - \xi_j}{\xi_j}\right) + \ln w_j = -\sum_{i=1}^{r} \lambda_i a_{ij} \quad \text{for} \quad j = 1, \ldots, n, \tag{8.5.1.3}$$

that is,

$$\xi_j = \frac{w_j}{w_j + \exp\left\{-\sum_{i=1}^{r} \lambda_i a_{ij}\right\}} \quad \text{for} \quad j = 1, \ldots, n. \tag{8.5.1.4}$$

Since $(\xi_1, \ldots, \xi_n) \in P(A,b)$, we also have

$$\sum_{j=1}^{n} \frac{a_{ij} w_j}{w_j + \exp\left\{-\sum_{i=1}^{r} \lambda_i a_{ij}\right\}} = b_i \quad \text{for} \quad i = 1, \ldots, r. \tag{8.5.1.5}$$

Equations (8.5.1.5) imply that $(\lambda_1, \ldots, \lambda_r)$ is the (necessarily unique) critical point of the strictly concave function

$$(s_1, \ldots, s_r) \longmapsto \sum_{j=1}^{n} \ln\left(1 + w_j \exp\left\{\sum_{i=1}^{r} a_{ij} s_i\right\}\right) - \sum_{i=1}^{r} b_i s_i,$$

and, consequently, $t_i^* = e^{\lambda_i}$ for $i = 1, \ldots, r$ is the point where the infimum in (8.5.1.1) attained. Hence the bound in the right hand side of (8.5.1.1) is

$$\exp\left\{-\sum_{i=1}^{r}\lambda_i b_i\right\}\prod_{j=1}^{n}\left(1+w_j\exp\left\{\sum_{i=1}^{r}a_{ij}\lambda_j\right\}\right), \tag{8.5.1.6}$$

while from (8.5.1.3) and (8.5.1.4), we get

$$\begin{aligned}
&\sum_{j=1}^{n}\left(\xi_j\ln\frac{1}{\xi_j}+(1-\xi_j)\ln\frac{1}{1-\xi_j}+\xi_j\ln w_j\right)\\
&=\sum_{j=1}^{n}\xi_j\left(\ln\frac{1-\xi_j}{\xi_j}+w_j\right)-\sum_{j=1}^{n}\ln(1-\xi_j)\\
&=-\sum_{j=1}^{n}\sum_{i=1}^{r}\xi_j\lambda_i a_{ij}+\sum_{j=1}^{n}\ln\left(1+w_j\exp\left\{\sum_{i=1}^{r}\lambda_i a_{ij}\right\}\right)\\
&=-\sum_{i=1}^{r}\lambda_i b_i+\sum_{j=1}^{n}\ln\left(1+w_j\exp\left\{\sum_{i=1}^{r}\lambda_i a_{ij}\right\}\right)
\end{aligned}$$

and hence the bounds (8.5.1.1) and (8.5.1.2) indeed coincide.

The advantage of (8.5.1.2) is that it admits a useful probabilistic interpretation. Let $X = (X_1, \ldots, X_n)$ be an n-vector of independent Bernoulli random variables such that

$$\mathbf{Pr}\left(X_j = 1\right) = \xi_j \quad \text{and} \quad \mathbf{Pr}\left(X_j = 0\right) = 1-\xi_j \quad \text{for} \quad j = 1, \ldots, n.$$

Then from (8.5.1.3) and (8.5.1.4), we conclude that for any vector $x \in \mathcal{X}_0(A, b)$, $x = (x_1, \ldots, x_n)$, we have

$$\begin{aligned}
\mathbf{Pr}\,(X = x) &= \prod_{j=1}^{n}\xi_j^{x_j}\left(1-\xi_j\right)^{1-x_j} = \prod_{j=1}^{n}\left(\frac{\xi_j}{1-\xi_j}\right)^{x_j}\prod_{j=1}^{n}(1-\xi_j)\\
&= \exp\left\{\sum_{j=1}^{n}\sum_{i=1}^{r}\lambda_i x_j a_{ij}+\sum_{j=1}^{n}x_j\ln w_j\right\}\prod_{j=1}^{n}\frac{1}{1+w_j\exp\left\{\sum_{i=1}^{r}\lambda_i a_{ij}\right\}}\\
&= \left(\prod_{j=1}^{n}w_j^{x_j}\right)\exp\left\{\sum_{i=1}^{r}\lambda_i b_i\right\}\left(\prod_{j=1}^{n}\frac{1}{1+w_j\exp\left\{\sum_{i=1}^{r}\lambda_i a_{ij}\right\}}\right).
\end{aligned}$$

We note that the probability that the random vector X hits a particular point $x \in \mathcal{X}_0(A, b)$ is proportional to the contribution $w_1^{x_1}\cdots w_n^{x_n}$ of that point x to the partition function $S_0(A, b; w)$. We obtain the following identity for the partition function:

$$\begin{aligned} S_0(A,b;w) =& \mathbf{Pr}\ (X \in \mathcal{X}_0(A,b)) \\ & \times \exp\left\{-\sum_{i=1}^{r}\lambda_i b_i\right\}\prod_{j=1}^{n}\left(1+w_j\exp\left\{\sum_{i=1}^{r}\lambda_i a_{ij}\right\}\right) \quad (8.5.1.7) \\ =& \mathbf{Pr}\ (X \in \mathcal{X}_0(A,b))\exp\left\{H_w\left(\xi_1,\ldots,\xi_n\right)\right\}. \end{aligned}$$

Comparing (8.5.1.6) and (8.5.1.7), we conclude that the upper bound (8.5.1.1)–(8.5.1.2) is just a consequence of the trivial bound

$$\mathbf{Pr}\ (X \in \mathcal{X}_0(A,b)) \ \le\ 1. \qquad (8.5.1.8)$$

One can try to improve the bound (8.5.1.1)–(8.5.1.2) by trying to strengthen (8.5.1.8). Thus we want to estimate the probability that a vector $X = (X_1,\ldots,X_n)$ of independent Bernoulli random variables satisfies the system of linear equations $\sum_{j=1}^{n} a_{ij}X_j = b_i$ for $i = 1,\ldots,r$. In [Sh10], Shapiro used anti-concentration inequalities to sharpen (8.5.1.8). In particular, one obtains

$$\mathbf{Pr}\ (X \in \mathcal{X}_0(A,b)) \ \le\ \min_{j_1,\ldots,j_r}\max\left\{\xi_{j_1},1-\xi_{j_1}\right\}\cdots\max\left\{\xi_{j_r},1-\xi_{j_r}\right\},$$

where the minimum of the products is taken over all collections of r linearly independent columns of the matrix A. This results in an improvement, often substantial, of the bound (8.5.1.1)–(8.5.1.2):

$$\begin{aligned} S_0(A,b;w) \ \le\ & \exp\left\{H_w\left(\xi_1,\ldots,\xi_n\right)\right\} \\ & \times \min_{j_1,\ldots,j_r}\max\left\{\xi_{j_1},1-\xi_{j_1}\right\}\cdots\max\left\{\xi_{j_r},1-\xi_{j_r}\right\}. \end{aligned}$$

Another useful observation is that

$$\mathbf{E}\left(\sum_{j=1}^{n}a_{ij}X_j\right) = \sum_{j=1}^{n}a_{ij}\xi_j = b_i \quad \text{for} \quad i = 1,\ldots,r.$$

Therefore, one may try to adapt the local Central Limit Theorem approach to estimate $\mathbf{Pr}\ (X \in \mathcal{X}_0(A))$ in (8.5.1.7). It is not hard to compute the $r \times r$ covariance matrix $Q = \left(q_{ij}\right)$ of random variables $\sum_{j=1}^{n} a_{1j}X_j,\ldots,\sum_{j=1}^{n} a_{rj}X_j$:

$$q_{ij} = \sum_{k=1}^{n} a_{ik}a_{jk}\left(\xi_k - \xi_k^2\right)$$

and the local Central Limit Theorem, when applicable, would imply that

$$\mathbf{Pr}\ (X \in \mathcal{X}_0(A,b)) \approx \frac{\det\Lambda}{(2\pi)^{r/2}\sqrt{\det Q}},$$

where $\det \Lambda$ is the determinant of the lattice $\Lambda \subset \mathbb{Z}^r$ generated by the columns of the matrix A. This approach was used in [BH13] to obtain asymptotically exact formulas for the number of graphs with a given degree sequence and for the number 0-1 integer matrices with prescribed row and column sums, and in [BH10], [Be14] for the number of 0-1 and non-negative integer d-dimensional arrays with prescribed sums over $(d-1)$-dimensional "slices".

Suppose that the set $\mathcal{X}_0(A,b)$ is not empty and let us consider it as a finite probability space, where $\mathbf{Pr}(x)$ is proportional to $w_1^{x_1}\cdots w_n^{x_n}$ for $x=(x_1,\dots,x_n)$. If there is a lower bound for $S_0(A,b;w)$, complementing the upper bound (8.5.1.1) and (8.5.1.2) as in Theorem 8.1.3 for the partition function of 0-1 flows, one can deduce that a random point $x \in \mathcal{X}_0(A,b)$ in many respects behaves as a vector $X=(X_1,\dots,X_n)$ of independent Bernoulli random variables. Indeed, it follows from (8.5.1.7) that $\mathbf{Pr}\ (X \in \mathcal{X}_0(A,b))$ is not too small, and hence various averaging statistics on X and $x \in \mathcal{X}_0(A,b)$ are sufficiently close. This observation was used in [B10b, BH13, C+11].

8.5.2 Partition functions of non-negative integer points. As in Sect. 8.5.1, let $A=\big(a_{ij}\big)$ be an integer $r\times n$ matrix of rank r, let $b=(b_1,\dots,b_r)$ be an integer r-vector, and let $w=(w_1,\dots,w_n)$ be a positive real vector of weights. We consider the set $\mathcal{X}_+(A,B)$ of non-negative vectors x that lie in the affine subspace defined by the system $Ax=b$:

$$\mathcal{X}_+(A,b)=\Big\{x=(x_1,\dots,x_n):\ \sum_{j=1}^n a_{ij}x_j=b_i \quad \text{for} \quad i=1,\dots,r \quad \text{and} \quad x_j\in\mathbb{Z} \quad \text{and} \quad x_j\ge 0 \quad \text{for} \quad j=1,\dots,n\Big\}.$$

To avoid convergence issues, we assume that $\mathcal{X}_+(A,b)$ is finite and consider a weighted sum (partition function) over $\mathcal{X}_+(A,b)$:

$$S_+(A,b;w)=\sum_{\substack{x\in\mathcal{X}_+(A,b)\\ x=(x_1,\dots,x_n)}} w_1^{x_1}\cdots w_n^{x_n}.$$

The estimates for $S_+(A,b;w)$ are similar to those for $S_0(A,b;w)$ in Sect. 8.5.1. Below, we briefly sketch them, see also [BH10]. We get an upper bound

$$S_+(A,b;w)\ \le \inf_{\substack{t_1,\dots,t_r>0\\ w_j t_1^{a_{1j}}\cdots t_r^{a_{rj}}<1\\ \text{for } j=1,\dots,n}} t_1^{-b_1}\cdots t_r^{-b_r}\prod_{j=1}^n \frac{1}{1-w_j t_1^{a_{1j}}\cdots t_r^{a_{rj}}}. \tag{8.5.2.1}$$

The dual form of (8.5.2.1) is as follows. Let $P_+(A,b)$ be the polyhedron that is the intersection of the non-negative orthant $\mathbb{R}^n_+$ with the affine subspace defined by the system $Ax=b$,

$$P_+(A,b) = \Big\{ x = (x_1, \dots, x_n) : \sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{for} \quad i = 1, \dots, r \quad \text{and} \quad x_j \geq 0 \quad \text{for} \quad j = 1, \dots, n \Big\}.$$

Suppose that $P_+(A,b)$ is bounded and has a non-empty relative interior, that is, contains a point $x = (x_1, \dots, x_n)$ such that $x_j > 0$ for $j = 1, \dots, n$. Let us consider a strictly concave function

$$G_w\left(x_1, \dots, x_j\right) = \sum_{j=1}^{n} \left((x_j + 1) \ln(x_j + 1) - x_j \ln x_j + x_j \ln w_j \right)$$
$$\text{where} \quad x_j \geq 0 \quad \text{for} \quad j = 1, \dots, n.$$

Then G_w attains its maximum on $P_+(a,b)$ at a unique point $(\xi_1, \dots, \xi_n)$ in the relative interior of P, see [BH10] for detail, and

$$S_+(A,b;w) \ \leq \ \exp\left\{ G_w\left(\xi_1, \dots, \xi_n\right) \right\}. \tag{8.5.2.2}$$

Moreover, the bounds of (8.5.2.1) and (8.5.2.2) are identical.

The probabilistic interpretation of the bound (8.5.2.1)–(8.5.2.2) is as follows. Let $X = (X_1, \dots, X_n)$ be an n-vector of independent geometric random variables such that

$$\mathbf{Pr}\left(X_j = k\right) = \frac{1}{1+\xi_j} \left(\frac{\xi_j}{1+\xi_j} \right)^k \quad \text{for} \quad k = 0, 1, \dots \quad \text{and} \quad j = 1, \dots, n,$$

so that $\mathbf{E}\, X_j = \xi_j$. Then

$$S_+(A,b;w) = \mathbf{Pr}\ \left(X \in \mathcal{X}_+(A,b)\right) G_w\left(\xi_1, \dots, \xi_n\right) \tag{8.5.2.3}$$

and (8.5.2.1), (8.5.2.2) can be written as

$$\mathbf{Pr}\ \left(X \in \mathcal{X}_+(A,b)\right) \ \leq \ 1.$$

In [Sh10], using anti-concentration inequalities, Shapiro obtained a stronger bound

$$\mathbf{Pr}\ \left(X \in \mathcal{X}_+(A,b)\right) \ \leq \ \min_{j_1, \dots, j_r} \frac{1}{\left(1+\xi_{j_1}\right) \cdots \left(1+\xi_{j_r}\right)},$$

where the minimum is taken over all collections $\{j_1, \dots, j_r\}$ of r linearly independent columns of A. The $r \times r$ covariance matrix $Q = \left(q_{ij}\right)$ of random variables $\sum_{j=1}^{n} a_{1j} X_j, \dots, \sum_{j=1}^{n} a_{rj} X_j$ is computed as

$$q_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk} \left(\xi_k + \xi_k^2 \right)$$

and the local Central Limit Theorem, when holds, implies that

$$\mathbf{Pr}\ (X \in \mathcal{X}_+(A,b)) \approx \frac{\det \Lambda}{(2\pi)^{r/2}\sqrt{\det Q}},$$

where $\Lambda \subset \mathbb{Z}^r$ is the lattice generated by the columns of A. In [BH12], this approach applied to obtain asymptotically exact formulas for the number of non-negative integer matrices with prescribed row and column sums.

Suppose that the set $\mathcal{X}_+(A,b)$ is not empty and let us consider it as a finite probability space, where $\mathbf{Pr}\,(x)$ is proportional to $w_1^{x_1} \cdots w_n^{x_n}$ for $x = (x_1, \ldots, x_n)$. If there is a lower bound for $S_+(A,b;w)$, complementing the upper bound (8.5.2.1) and (8.5.2.2) as in Sect. 8.4 for the partition function of integer flows, one can deduce that a random point $x \in \mathcal{X}_+(A,b)$ in many respects behaves as a vector $X = (X_1, \ldots, X_n)$ of independent geometric random variables. Indeed, it follows from (8.5.2.3) that $\mathbf{Pr}\ (X \in \mathcal{X}_0(A,b))$ is not too small, and hence various averaging statistics on X and $x \in \mathcal{X}_0(A,b)$ are sufficiently close. This observation was used in [B10a].

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Index

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